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	Page / Страница	Line / Строка	Designated / Напечатано	Correction / Должно быть
*	**	from top / сверху / from bottom / снизу		
20	341	7 сверху	$\dots (1/\beta_0 n))$	$\dots (1/\beta_0 n)]$
91	380	16 снизу	$w_s = \frac{\omega_0^2}{Q^2} \dots$	$w_s = \frac{\omega_0^2}{Q_l^2} \dots$
94	382	14 сверху	$\dots (r-m+1)(\ +m) \dots$	$\dots (r-m+1)(r+m) \dots$
108	391	4 снизу	$\dots \overline{(\Delta\psi)^2_{r_1}}/\overline{(\Delta\psi)^2_{r_1}} = 10$ км	$\dots \overline{(\Delta\psi)^2_{r_1}}/\overline{(\Delta\psi)^2_{r_1=10\text{ км}}}$
116	396	10 снизу	$\dots (1,\ rr_1^{-1} - 0{,}5) \dots$	$\dots (1{,}5\, rr_1^{-1} - 0{,}5) \dots$
164	430	10 снизу	$\langle \Delta h \rangle C_h^2 \ll 1$	$\langle \Delta h^2 \rangle C_h^2 \ll 1$
185	444	16 снизу	$\Gamma\varphi_\rho = -\frac{1}{\varepsilon}\rho(xyz,\ t)$	$\Delta\varphi_\rho = -\frac{1}{\varepsilon}\rho(xyz,\ t)$
190	447	1 сверху	$+ V_s^-(z)\, e^{-ih_{0s}z}$	$+ V_s^-(z)\, e^{ih_{0s}z}$
265	495	3 снизу	$(r_k - l_k)\,2$	$(r_k + l_k)/2$
293	512	3 сверху	e^{-xpy}	$e^{-x_p y}$

*[translation page]
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RADIO PHYSICS SERIES

[Following is the complete translation of the journal entitled "Izvestiya Vysshikh Uchebnykh Zavedeniy, MVO, Radiofizika (English version above), Vol. 2, No. 3, 1959, pages 351-512.]

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ON SOME PECULIARITIES OF ELECTROMAGNETIC WAVES RADIATED BY PARTICLES MOVING FASTER THAN LIGHT*.

Pages 331-343

By V.L. Ginzburg & V.I. Eidman

In analysing peculiarities in the radiation of electromagnetic waves by particles or systems (atoms, molecules, etc.) moving in a medium or close to a medium (a retarding system) with speeds faster than light, quantum representation is often found fruitful, even in classical problems. The same is true in examining the radiation, absorption and amplification of waves in a beam of particles. The present paper evaluates a number of problems, among which is that of the classical radiation reaction observed in the motion of a charge in a medium, or close to a medium (in a channel, a slot, etc.) The basic conclusion, drawn from simple quantum considerations as well as from quantitative classical calculations is as follows: in the case of movement at speeds faster than that of light, the radiation reaction which modifies the oscillation amplitude of the particles is reduced in comparison with the case of movement at speeds below that of light and in an anisotropic medium it may even change sign, i.e. correspond not to "friction", but to a build-up of the oscillations. This effect is directly related to the instability of beams of particles at speeds greater than light.

It is known that when particles (electron, proton) or "systems" (an atom, a plasma cluster an antenna) move in, or near a medium at speeds greater than that of light, we get the Vavilov - Cerenkov effect and the character of the Doppler effect is substantially altered.

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* Lecture at the MVO Radioelectronics Conference, Kiev, 1959

Classical reasoning [1] leads, in the case of the Vavilov-Cerenkov effect, to the radiation condition

$$\cos\theta_0 = c/n(\omega)\,v, \qquad ; \qquad (1)$$

θ_0 is the angle between the particle velocity v and wave vector $\underline{k}$ of the Cerenkov waves; $n(\omega)$ is the refractive index for the frequency ω under consideration. The medium is considered to be isotropic.

Condition (1) has a kinematic (interference) character and is therefore kept independently of the form of the emitter (charge, dipole, etc); it is also found in an anisotropic medium [2], but with $n(\omega)$ replaced by $n_j(\omega, \underline{k}/k)$, which is the refractive index for a normal wave of the type $j = 1,2$, propagated in the direction $\underline{k}$ (in the case of movement along the axis of a uniaxial crystal or along a magnetic field in a plasma $n_j = n_j(\omega, \theta_0)$ and (1) is also the equation for determining $\theta_0(\omega)$).

For a point charge, moving uniformly in an isotropic medium, the energy radiated in unit time due to the Vavilov - Cerenkov effect is [1]:

$$\left(\frac{dW}{dt}\right)_{ep} = \frac{e^2 v}{c^2} \int_{c/n(\omega)<v}^{\infty} \left[1 - \frac{c^2}{v^2 n^2(\omega)}\right] \omega\, d\omega. \qquad (2)$$

However, if an emitter, which in its own frame of reference emits waves with a frequency ω_0, moves (in a medium, then, in the frame of reference connected with the medium,) the radiation due to the Doppler effect will have a frequency $\omega(\theta)$:

$$\omega = \frac{\omega_0 \sqrt{1-\beta^2}}{|1-\beta n(\omega)\cos\theta|}; \quad \beta = \frac{v}{c}. \qquad (3)$$

When $\beta n < 1$, i.e. if the speed is slower than that of light, formula (3) constitutes a long known expression for the Doppler effect in a medium. However, if $\beta n > 1$ (at speeds faster than that of light it is necessary to introduce a modulus [3] into

(3) and it is advisable to examine separately the radiation inside and outside the Cerenkov cone*. Inside the cone (when $\theta < \theta_0$), the Doppler effect is anomalous, because the frequency increases with increase in θ, and if n = const, then $\omega \to \infty$ when $\theta \to \theta_0$. The Doppler effect outside the cone can of course be called normal, since the frequency ω decreases with increase in θ (see Fig. 1).

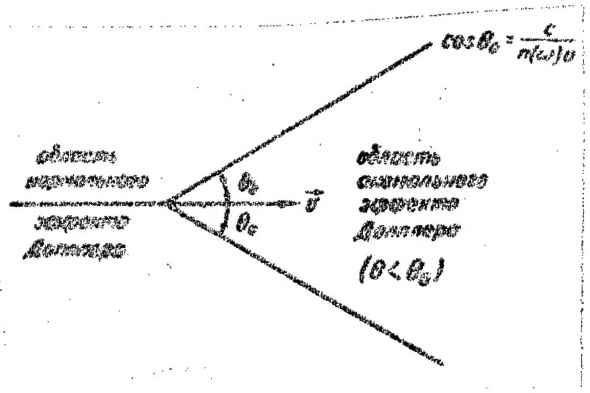

Fig. 1

1. Region of normal Doppler effect
2. Region of anomalous Doppler effect

Formula (3) reflects the general characteristic of radiation in a medium; the role of the quantity $\beta = v/c$ for a vacuum is played by the quantity βn in the case of a medium. Therefore, if in a vacuum the extreme relativistic case corresponds to the value $\beta \to 1$, then in a medium, from the point of view of the character of the radiation the extreme relativistic case corresponds to the value $\beta n \to 1$. Moreover, if radiation in a vacuum when $\beta \to 1$, is sharply directed along the velocity, then

* In the case of an anisotropic medium (and also in the presence of gyrotropy), it is only necessary to exchange $n_j(\omega)$ in (3) $n_j(\omega, \underline{k}/k)$. In the text, unless the contrary is stated, it is assumed for the sake of simplicity that the medium is isotropic.

in a medium the role of preferred direction is played by the Cerenkov cone. However if we are dealing, for example, with a medium in which $n < 1$ (isotropic plasma), then at no velocity $\beta < 1$ does the radiation possess the features which are characteristic of the extreme relativistic case in a vacuum.

It should also be mentioned that the practical value of the Doppler effect in a medium as a result of the large losses during the motion of an emitter in a dense body is connected with the following two factors. In the first place the characteristic peculiarities of the Doppler effect at speeds faster than that of light are also preserved in motion through narrow channels and slots in a medium [4, 5] or close to a medium or an artificial decelerating system. Secondly, of even greater interest is motion in a magnetically active plasma [6 7], when the losses are small. The above also pertains of course to Cerenkov radiation, although in this case, as is well known, the effect can also be observed when charges move in a dense, continuous medium.

1. The analysis of various problems connected with the absorption and amplification of electromagnetic waves during the movement of charges or various "systems" in a medium is greatly helped by elementary quantum considerations. This also applies to cases where the problem is essentially of a classical nature and consequently, given the accuracy assumed, the final formulas do not depend on the quantum constant $\hbar$.

The starting concept in applying quantum considerations is the idea of quanta in a medium, the energy of which equals $\hbar\omega$ and the momentum $\hbar\underline{k} = (\hbar\omega/c)\, n(\omega, s)s$, where $\underline{k} = k\underline{s}$ is the wave vector and n is the refractive index for a normal wave of the given type, propagated in the medium under consideration(in the general case, anisotropic and gyrotropic). In the

case of an anisotropic medium the quantization has been made in [8]; generalization of the result to apply to any arbitrary medium is readily obtained by plane wave expansion [9, 10]. It is understood that such an approach is correct only in the region of applicability of phenomenological theory. It should also be remembered that in this case the momentum of the quantum in a medium is a total momentum, comprising both the field momentum and the momentum connected with the motion of the medium in the field of the wave (see [11]).

From the quantum point of view, of the radiation kinematics i.e. the conditions applying to frequencies and direction of radiation, are determined by the laws of conservation of energy and momentum (the same applies to the absorption conditions. If, for example, prior to emission the "system" (electron, atom, antenna) had an energy E_0 and after emission its energy is E_1 and the corresponding momenta are p_0 and p_1, then in the emission of a quantum the conservation laws:

$$E_0 - E_1 = \hbar\omega; \tag{4}$$

$$p_0 - p_1 = \hbar k = \frac{\hbar\omega n}{c}\frac{k}{k} = \frac{\hbar\omega n s}{c}. \tag{5}$$

should be observed.

For a "system " moving uniformly in a vacumm (i.e. when $n = 1$), emission without a change in the internal state of the "system" is impossible (for example, an electron moving uniformly in a vacuum cannot emit). This well known fact understandably follows from (4) - (5), since when $n = 1$ these equations for particles without internal degrees of freedom only have the solution $\omega = 0$. In the general case (when $n \neq 1$), substituting

in (4) - (5) $E_{o,1} = \sqrt{m^2c^4 + c^2p_{o,1}^2}$, $E_{o,1} = mv_{o,1}/\sqrt{1-v_{o,1}^2/c^2}$, as the condition for emission we get [8]:

$$\cos\theta = \frac{c}{n(\omega)v_0}\left[1 + \frac{\hbar\omega(n^2-1)}{2mc^2}\sqrt{1-v_0^2/c^2}\right]; \qquad (6)$$

$$\hbar\omega = \frac{2(mc/n)(v_0\cos\theta - c/n)}{\sqrt{1-v_0^2/c^2}(1-1/n^2)}.$$

At $\hbar\omega/mc^2 \ll 1$ this condition becomes the classical one (see (1)), which is natural (if $\hbar\omega/mc^2 \ll 1$, then the "output" connected with the emission of a quantum is small)*. It is clear from (6) that emission is possible (i.e. $\cos\theta \leq 1$ and $\omega > 0$) only when $v_0 n/c = \beta n > 1$.

When the result does not contain $\hbar$, the quantum calculation has only a methodological value. Essentially, the question reduces to the utilization of conservation laws which have a wider significance in the sense that they can also be used without introducing quantum considerations. Let us consider that the ratio $\frac{\mathcal{E}}{P} = (n/c)v$ is obtained from the classical theory of an electromagnetic field in a medium as the relationship between the energy $\mathcal{E}$ and the total momentum of the radiation and the medium P. Further, for free motion of the charge $\Delta E = v\Delta p$, since

$$\frac{dE}{dp} = \frac{d}{dp}\left(\sqrt{m^2c^4 + c^2p^2}\right) = \frac{c^2p}{E} = v;$$

* If n is very large, then, as is clear from (6), classical condition should be written somewhat differently, namely in the form

$$(\hbar\omega n^2/2mc^2)\sqrt{1-v_0^2/c^2} \ll 1.$$

hence, and from the conservation laws (4) - (5) (exchanging $\hbar\omega$ for $\mathcal{E}$) $\Delta E = \mathcal{E} = v\Delta p = \mathcal{E}(n/c)sv$ or $\cos\theta_o = c/nv$, i.e. condition (1) is obtained. However, the direct introduction of quanta $\hbar\omega$ is simpler** and quite natural not only in the quantum but also in the classical case. We shall proceed accordingly.

If a "system", not a particle, is in motion and the internal energy of this system can change, then $E_o = \sqrt{(m + m_o)c^2 + c^2p_o^2}$, $E_1 = \sqrt{(m + m_1)^2 c^4 + c^2p_1^2}$, where $(m + m_o)c^2 = mc^2 + \mathcal{E}_o$ is the total energy in the initial state and $(m + m_1)c^2 = mc^2 + \mathcal{E}_1$ is the total energy in the final state. It is obvious that $\mathcal{E}_1 - \mathcal{E}_o = \hbar\omega_o$ is the difference in energies of the two levels of the "system" (atom, etc.) under consideration.

Applying the conservation laws (4) - (5), when $\hbar\omega/mc^2 \ll 1$ we obtain [12] the Doppler condition (3). But at the same time light is thrown on a very important circumstance which completely escapes consideration in the classical derivation [3] of formula (3). In the region of the normal Doppler effect, i.e. when

$$\beta n(\omega)\cos\theta < 1, \qquad (7)$$

emission corresponds to transition of the system from a higher level with the energy $\mathcal{E}_1$ to a lower level with the energy

---

**Suffice it to say that the relationship $\underline{P} = \mathcal{E}(n/c)\underline{s}$ follows directly on quantization [8], while in the classical theory of a field this problem is quite complex and has been finally cleared up [11] only recently.

$\mathcal{E}_0 < \mathcal{E}_1$ (the direction of transition is determined from the requirement that the energy of the emitted quantum be positive i.e., formally, from the requirement $\omega > 0$). However, if the quantum is emitted inside the Cerenkov cone, i.e. if an anomalous Doppler effect is present:

$$\beta n \cos\theta > 1, \qquad (8)$$

emission of the quantum is accompanied by a transition of the "system" from the lower level $\mathcal{E}_0$ to the higher one $\mathcal{E}_1$. The energy of a quantum, as well as the energy used in exciting the emitting "system" is then drawn from the kinetic energy of progressive motion of the "system".

It is evident from this example, that in quantum theory, as distinct from the classical, finding the conditions of emission gives at the same time the direction of the process (transition up or down). This circumstance, together with the possibility of simply accounting for induced emission (see below), constitutes the value of quantum calculations for obtaining the conditions of emission and amplification (instability) of waves in a flux of particles, etc.

If the "system" has only two discrete levels, 0 and 1, then when $\beta n < 1$ (motion at speeds less than that of light) stationary state of the emitter corresponds to its being at the lower level 0 (it is assumed that the "system" is moving, let us say in a canal in a medium and that there are no extraneous sources of excitation). In other words, if level 1 is excited, then within a certain time the "system" will be de-excited with a transition to the level 0. However, if $\beta n > 1$ (motion at speeds faster than that of light), then it is probable that the "system" will be found at level 1 even under stationary conditions, while continuously radiating both normal and anomalous Doppler

waves. The population of levels 1 and 2 and the intensity of emission of the normal and anomalous waves are evidently determined by the ratio of total probabilities of emission of these waves. In the case of a "system" with many levels [13] emission of anomalous Doppler waves with an upward shift of the system can result in the building of "transverse oscillations" and, for instance, ionization of the atom. More precisely, two cases are possible here [14]. In the first case the mean energy of the transverse oscillations of the "system" decreases with movement. This means, that for a wave-packet, composed of wave functions with different, but nearly equal energies (for example, the energies of transverse oscillatory motion of an electron moving along a magnetic field), the center of gravity of the packet in the energy scale is depressed. In this case the difference between speeds less than and greater than that of light consists in the different rate of change of mean energy and also in the character of the spreading of the packet. Thus at speeds slower than that of light, the states with an energy greater than that represneted in the initial spectrum of the packet will never prove to be occupied. However, at speeds faster than that of light, a decrease in mean energy notwithstanding, there is a finite probability of finding a "system" (assuming of course, that we are dealing with an ensemble of systems) on any level however high, attainable while observing condition (8). In the second case of motion at speeds faster than that of light, the "system" is already unstable "on the average", i.e. its mean energy increases with time, not to mention the spreading of the packet.

The decision as to which of these possibilities we are dealing with requires concrete calculations of transition probabilities. Here, generally speaking, the application of

quantum calculations offers no advantage and it is natural to use classical theory. We shall return to this question later.

Quantum considerations of the type adduced above are no less useful in the analysis of the previously mentioned problems of absorption and amplification of waves in a flux of particles. In the case of wave amplification the flux, in essence, becomes unstable. In this way it is easy to obtain a criterion of the instability of a flux of particles moving in an isotropic plasma [15]. Furthermore, it is clear that in the case of motion at speeds faster than that of light of a flux "systems" with two or more levels, amplification (negative absorption), and not absorption (reabsorption of anomalous Doppler waves [16] should generally take place. This is connected with the fact that in the absorption of a quantum from the region of anomalous Doppler waves (i.e. one travelling at an angle $\theta < \theta_0$ to the velocity of the "system"), the transition of the "system" will be not from below up (as in the case of a normal effect), but from above-down*. However, a transition of the "system" from below up now corresponds to induced emission, which in the region of normal effects corresponds to a transition of the "system" from above down. Therefore, if all the "systems" (atoms, electrons in a magnetic field) in a faster - than - light beam are on the lower level, for example, normal Doppler waves emitted by one of the systems will be absorbed in this beam and anomalous waves amplified and they will translate other systems in their path upwards from below with induced emission, i.e. with the emission

* Absorption is a process opposite to emission and therefore the above follows directly from calculations made for emission.

of one more anomalous Doppler quanta. If both the upper and the lower (0 and 1) levels are occupied then the coefficient of absorption in the beam for normal Doppler waves is equal (see [15, 16]) to

$$\mu_{\omega} = \frac{dI_{\omega}}{I_{\omega}} = A_1^0 \frac{8\pi^3 c^2 N_1 (N_0/N_1 - 1)}{\omega^2 n^2}, \qquad (9)$$

where $A_1^0(\Theta)$ is the probality of a trnasition $1 \leftrightarrow 0$, referred to a unit solid angle, with the emission of a quantum at an angle Θ to the velocity; N_1 and N_0 are the concentrations of particles in the beam at the levels 1 and 0 respectively n is the refractive index of the medium for the frequency ω under consideration when the waves are propagated at an angle Θ (for the sake of simplicity we shall consider that the dipole moment for the trnasition $1 \leftrightarrow 0$ for all the particles is parallel to the velocity). If amplification of normal Doppler waves is to take place, the number of particles at the upper level 1 should exceed the number at the lower level 0 (in this case $N_0/N_1 < 1$ and $\mu < 0$). A similar distribution according to levels is not found at thermal equilibrium and its creation is generally connected with certain difficulties. The situation changes in the case of anomalous Doppler waves, in which emission of waves (quantum) occur with the transition $0 \rightarrow 1$, and absorption with the transition $1 \rightarrow 0$. In this case

$$\mu_{\omega} = A_1^0 \frac{8\pi^3 c^2 N_0 (N_1/N_0 - 1)}{\omega^2 n^2} \qquad (10)$$

and $\mu < 0$ when $N_1/N_0 < 1$. Hence it is clear that in the presence of an anomalous Doppler effect (i.e. if $\beta n(\omega) > 1$) a beam of particles at just one lower level 0 has negative absorption and the waves emitted by the individual particles are amplified. This circumstance appears to be very favorable from the point of view

of the possibility of utilizing beams of particles, moving in a dielectric slot or a decelerating system, for the generation and amplification of microwaves [17].

As mentioned previously the role of a system (particle) emitting anomalous Doppler waves can be played by an electron oscillating under the influence of an applied field or by one moving along a helical path down a magnetic field parallel to the axis of the beam. At small amplitudes such electrons, neglecting Cerenkov radiation, emit in a manner similar to corresponding oscillators moving with a velocity v equal to the projection parallel to the axis of the beam of the electron velocity $v_{\parallel}$.

The transverse velocities $v_{\perp}$ in an electron flux are usually distributed so that the distribution function $f(v_{\perp})$ decreases with increase in $v_{\perp}$ (this occurs, for instance, for a distribution $f(v_{\perp}) = \text{const} \exp(-m v_{\perp}^2/2xT)$). Under similar conditions normal Doppler waves will be attenuated as a result of reabsorption in the beam; on the other hand, anomalous Doppler waves are amplified. Amplification of waves in an electron flux means that the amplitude of the oscillations increases and the flux loses stability. In such cases, generally speaking, a grouping of electrons takes place and we get coherent emission. The quantum condition for instability of the flux (the condition $\beta n(\omega) > 1$)* coincides with the condition which can be obtained by solving the classical problem of the stability of an electron flux in a magnetic field [16]. This instability of electron fluxes, which is encountered in particular in magnetic-

* For the sake of simplicity, we limit ourselves to the case where the velocity of all the electrons along the field is the same and is equal to $v = \beta c$ for fast electrons and for small amplitudes of transverse motion the difference between v and $v_{\parallel}$ is also unimportant; for details see [16].

ally active plasmas, is of interest from the point of view of the theory of sporadic solar radio emission [18].

To conclude this section, let us note an interesting point connected with the non-coincidence of the directions of the phase and group wave velocities, which may occur in an anisotropic medium or when spatial dispersion is taken into account* . If the projection of group velocity $d\omega/d\underline{k}$ on the direction perpendicular to the particle velocity (i.e. the quantity $d\omega/dk_r$, where k_r is the projection of $\underline{k}$ perpendicular to v) is negative, then the energy, it would seem, is not given out by the source, but is absorbed by it. However, under these conditions one must utilize not the retarded, but the advanced potentials [19, 21]. If the vector $\underline{k}$ is always chosen in the direction of the phase velocity, then when $d\omega/dk_r < 0$ this vector $\underline{k}$ will be directed towards the particle trajectory in both Cerenkov and Doppler waves and energy, as it should, is given out from the trajectory. In the case of Cerenkov emission the difference between the cases $d\omega/dk_r > 0$ and $d\omega/dk_r < 0$ is clear from Fig. 2. For $d\omega/dk_r < 0$ the angle θ_0 is first determined by condition (1), as is clear from the choice of the direction of $\underline{k}$ from interference considerations and from the conservation laws (4) - (5). The foregoing follows from the fact that we are using plane waves of the type $e^{i(\omega t - \underline{kr})}$ for which the momentum if the corresponding quantum in a medium is equal to $(\hbar\omega n/c)\underline{k}/k$; using such waves there is no difference between the directions of $\underline{k}$ in Figs. 2a and 2b, since in terms

* See [19] and also [20] (Figs. 1 and 2 in [20] show functions $n^2(\omega)$ for which in certain regions $d(n\omega)/d\omega < 0$ and, consequently for which the signs of the phase and group velocities are different).

of plane waves the disposition of the wave fronts is identical in both cases

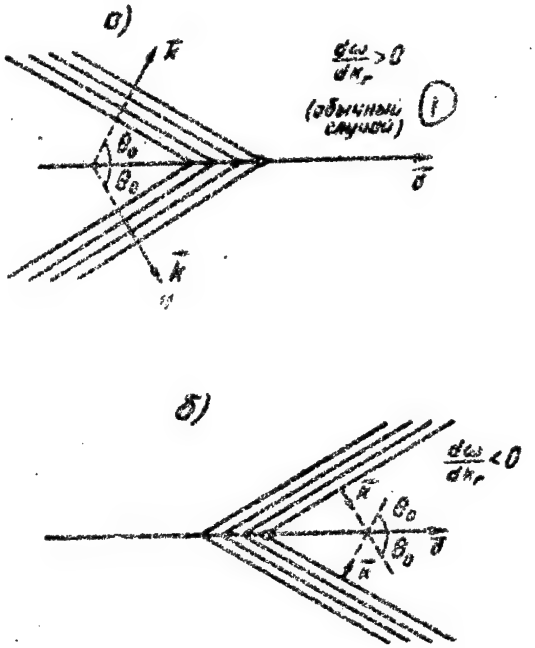

Fig. 2

① Top (ordinary case)

(we have in mind all wave fronts with vector $\underline{k}$, lying on the Cerenkov cone)**. Formula (3) when $d\omega/dk_r < 0$) also remains valid.

** It is understood that if vector $\underline{k}$ is always directed away from the particle, then at $d\omega/dk_r < 0$ the angle between $\underline{k}$ and $\vec{v}$ will be determined by the condition $\cos\theta_0 = -c/nv$ (the wave front pattern does not depend on the choice of the sign of $\underline{k}$). The choice of vector $\underline{k}$ used in the text is more consistent, since it corresponds to the usual relationship between $\underline{k}$ and the direction of phase velocity.

2. For electrons moving in a magnetic field at speeds faster than that of light in a plasma or in a decelerating system and also in analogous cases of oscillatory motion of electrons usually only the classical region is of interest (in this case large quantum numbers correspond to transverse motion). In such cases the problem of the radiation of waves and of the damping or build-up of transverse electron oscillations may and in practice must be solved by classical calculations. Essentially, these calculations reduce to finding the radiation reaction when a charge moves in a medium. Let us examine this problem in somewhat greater detail.

Since the presence of a medium can radically alter the character of emission of electromagnetic waves by moving particles, it is obvious that the radiation reaction in a medium will also be changed, sometimes in a most substantial manner. As an illustration, we may point to the fact that an oscillator with a frequency ω in an isotropic medium with a refractive index $n = \sqrt{1 - 4\pi e^2 N/m\,\omega^2}$ does not in general emit for $\omega_0^2 = 4\pi e^2 N/m > \omega^2$, when $n^2 < 0$; in a magnetic plasma, in a non-relativistic appriximation, there is no emission from an electron rotating in a magnetic field H_0 with a frequency $\omega_H = eH_0/mc$ (see [23]). In both these cases the radiation force will obviously vanish, whereas in a vacuum it will be equal to

$$f_0 = \frac{2e^2}{3c^3}\frac{d^2v}{dt^2}; \qquad (11)$$

On the other hand, in the case of uniform motion in a medium, if for certain frequencies the velocity $v > c/n(\omega)$, we get a radiation force f_{rad}, the work done by which in unit time

$f_{\text{тр}} v = dW_{\text{тр}}/dt$. Therefore it is evident from (2) that

$$f_{\text{тр}} = -\frac{e^2 v}{c^2 v} \int_{c/n < v} \left[1 - \frac{c^2}{v^2 n^2(\omega)} \right] \omega \, d\omega. \qquad (12)$$

In view of the above, there arises the problem of calculating the radiation reaction in the case of arbitrary motion of a charge in an arbitrary medium. However, to the best of our knowledge, this problem has so far attracted no attention. Apparently this is due to the fact that the radiation force for motion in a medium is considerably smaller than the retarding force, connected with ionizing losses. Thus, the losses for Cerenkov emission which could be considered radiation losses, even in a transparent but dense medium, constitute only a small portion of the total loss. The situation, generally speaking, is not changed in the case of non-uniform motion. However, as was said before, there exist interesting and practically important cases in which the inclusion of the radiation forces for motion in a medium is essential (motion in canals, slots and near a medium, motion in a magnetically active plasma).

An expression for the radiation reaction in a medium has been obtained in [14]. Here we shall introduce only a brief survey of the calculations. For a point charge, the field equations and the equation of motion have the form:

$$\operatorname{rot} \boldsymbol{H} = \frac{4\pi}{c} e\boldsymbol{v}\delta(\boldsymbol{r} - \boldsymbol{R}) + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}; \quad \operatorname{div} \boldsymbol{D} = 4\pi e\delta(\boldsymbol{r} - \boldsymbol{R}); \qquad (13)$$

$$\operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}; \quad \operatorname{div} \boldsymbol{H} = 0;$$

$$\frac{d}{dt}\left(\frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \right) = e\left\{ \boldsymbol{E}^{(0)} + \frac{1}{c}[\boldsymbol{v}\boldsymbol{H}^{(0)}] \right\} + e\int \left\{ \boldsymbol{E}(\boldsymbol{r}) + \right.$$

$$\left. + \frac{1}{c}[\boldsymbol{v}\boldsymbol{H}(\boldsymbol{r})] \right\} \delta(\boldsymbol{r} - \boldsymbol{R})\, d\boldsymbol{r}. \qquad (14)$$

Where $\underline{R}(t)$ is the radius vector of the position of the charge ($v = d\underline{R}/dt$); $\underline{E}^{(0)}$ and $\underline{H}^{(0)}$ are the external fields and $\underline{E}$ and $\underline{H}$ fields formed by the charge itself (the medium is considered non-magnetic for the sake of simplicity).

In the case of an arbitrary medium, the only effective solution of the problem is to expand the fields into normal plane waves [7, 9, 10]. As a result we have:

$$\tilde{D}_\alpha^{(\omega)} = \varepsilon_{\alpha\beta}(\omega)\,\tilde{E}_\beta^{(\omega)} \qquad (\alpha,\beta = 1,2,3); \tag{15}$$

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\varphi; \quad H = \operatorname{rot} A;$$

$$\tilde{A} = \sqrt{4\pi}\, c \sum_{\lambda, j=1,2} \frac{q_{\lambda j}(t)\, a_{\lambda j}}{n_{\lambda j}}\, e^{i k \lambda r}; \tag{16}$$

$$\varepsilon_{\alpha\beta}\frac{\partial \tilde{A}_\alpha}{\partial x_\beta} + \text{к.с.} = 0, \tag{17}$$

where the summation is over the indices; which occur twice the index ω shows that Fourier components are taken and the material fields are $\underline{D} = \tilde{\underline{D}} + \tilde{\underline{D}}^*$, $\underline{E} = \tilde{\underline{E}} + \tilde{\underline{E}}^*$, etc. In equations (15) - (17) $n_{\lambda j}$ is the refractive index and $\underline{a}_{\lambda j}$ is the complex polarisation vector, corresponding to the j-th normal wave. The following are the equations for potentials, obtained from (13), (15), (17):

$$\Delta\tilde{A} - \operatorname{grad}\operatorname{div}\tilde{A} - \frac{1}{c^2}\varepsilon_{\alpha\beta}\frac{\partial^2 \tilde{A}_\beta}{\partial t^2}\, e_\alpha - \frac{1}{c}\varepsilon_{\alpha\beta}\frac{\partial^2 \tilde{\varphi}}{\partial t\, \partial x_\beta}\, e_\alpha + \text{к.с.} = -\frac{4\pi}{c} e\omega\delta(r - R);$$

$$\varepsilon_{\alpha\beta}\frac{\partial^2 \tilde{\varphi}}{\partial x_\alpha\, \partial x_\beta} + \text{к.с.} = -4\pi e\,\delta(r - R), \tag{18}$$

where e_α is the unit vector of the α - axis.

The substitution of (16) in (18) gives a system of oscillator equations for amplitudes of the field $q_{\lambda j}$. Its solution is elementary. The fields determined in this way must be substituted in the equation of motion (14). As a result, we obtain (see [14]:

$$\frac{d}{dt}\left(\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}\right)=\boldsymbol{F}^{(0)}-\frac{e^2}{2\pi^2}\sum_{j=1,2}\int_0^t\int_0^{\omega_{\text{макс}}}\left\{\frac{\boldsymbol{a}_j(\boldsymbol{v}'\boldsymbol{a}_j^*)}{n_j^2}\cos\omega_j(t-t')-\right. \tag{19}$$

$$\left.-i\,[\boldsymbol{v}\,[\boldsymbol{k}\boldsymbol{a}_j]]\,\frac{(\boldsymbol{v}'\boldsymbol{a}_j^*)}{n_j^2\omega_j}\sin\omega_j(t-t')\right\}e^{i\boldsymbol{k}(\boldsymbol{R}-\boldsymbol{R}')}\,dt'\,d\boldsymbol{k}+\text{к.с.}=\boldsymbol{F}^{(0)}+\boldsymbol{f}_p,$$

where $\boldsymbol{F}^{(0)}=e\{\boldsymbol{E}^{(0)}+c^{-1}[\boldsymbol{v}\boldsymbol{H}^{(0)}]\}$.

The method adopted for the calculation of the radiation reaction is convenient for a number of cases even in an isotropic medium or in a vacuum. Thus, for the radiation friction, acting on a particle moving in a vacuum with a non-relativistic velocity, expression (11) is easily obtained from (19) (see also [24, 25]. On the other hand, for particles moving uniformly in an isotropic medium with a refractive index $n > c/v$, formula (19) yields a formula for the retarding force of Cerenkov radiation.

An examination of an oscillator moving at speeds faster than that of light has been made in [14]. For an oscillator in an isotropic medium, oscillating parallel to the progressive velocity v_0, we have:

$$\boldsymbol{R}=\{0;0;v_0t+R_0\sin\Omega t\};\quad \boldsymbol{v}=\{0;0;v_0+v_\sim\cos\Omega t\};$$
$$v_\sim=R_0\Omega; \tag{20}$$
$$\boldsymbol{a}_1=\{1;0;0\};\ \boldsymbol{a}_2=\{0;\cos\Theta;-\sin\Theta\};\ \boldsymbol{k}=\{0;\ k\sin\Theta;\ k\cos\Theta\},$$

while below we shall be dealing only with the case in which

$$kR_0 = (\omega/c)\, n(\omega) R_0 \ll 1. \tag{21}$$

Under similar conditions we obtain from (19) the following expression for the work done by the radiation field on the particle:

$$A = \int_0^T v f_p\, dt = v_0 \int_0^T f_{pz}\, dt + v_- \int_0^T \cos \Omega t\, f_{pz}\, dt = A_0 + A_-; \tag{22}$$

$$A = -\frac{e^2 R_0^2 T}{4c^3 \beta_0} \left\{ \int\limits_{\beta_0 n(\omega)\cos\Theta < 1} \omega^3 \left[1 - \frac{1}{\beta_0^2 n^2(\omega)} \left(1 - \frac{\Omega}{\omega} \right)^2 \right] d\omega + \right.$$
$$\left. + \int\limits_{\beta_0 n(\omega)\cos\Theta > 1} \omega^3 \left[1 - \frac{1}{\beta_0^2 n^2(\omega)} \left(1 + \frac{\Omega}{\omega} \right)^2 \right] d\omega \right\}, \tag{23}$$

where

$$\omega = \frac{\Omega}{|1 - \beta_0 n(\omega) \cos\Theta|}; \qquad \beta_0 = \frac{v_0}{c}. \tag{24}$$

If the dispersion dependence is given by the following relationships (see [14]):

$$\begin{aligned} n(\omega) &= n = \text{const} && \text{when} && \omega < \omega_c; \\ n(\omega) &= 1 && \text{when} && \omega > \omega_c, \end{aligned} \tag{25}$$

then (23) can be written in the form:

$$A = -\frac{e^2 \Omega^4 R_0^2 n T}{4c^3} \int \frac{\sin^3\Theta\, d\Theta}{|1 - \beta_0 n \cos\Theta|^3}, \tag{26}$$

where, for the anomalous Doppler effect

$$0 \leqslant \theta \leqslant \arccos\left[\frac{1}{\beta_0 n}\left(1+\frac{\Omega}{\omega_c}\right)\right],$$

and for the normal Doppler effect,

$$\arccos\left[\frac{1}{\beta_0 n}\left(1-\frac{\Omega}{\omega_c}\right)\right] \leqslant \theta \leqslant \pi.$$

The quantity $W = -A$ is equal to the energy radiated by the particle (see [3]). The work of the radiation field, expended on increasing or diminishing the oscillation of the particle is, in agreement with (22),

$$A_\sim = \frac{e^2 \Omega R_0^2 T}{4c^3 \beta_0}\left\{\int\limits_{\beta_0 n(\omega)\cos\theta > 1} \omega^2\left[1-\frac{1}{\beta_0^2 n^2(\omega)}\left(1+\frac{\Omega}{\omega}\right)^2\right]d\omega - \right.$$
$$\left. - \int\limits_{\beta_0 n(\omega)\cos\theta < 1} \omega^2\left[1-\frac{1}{\beta_0^2 n^2(\omega)}\left(1-\frac{\Omega}{\omega}\right)^2\right]d\omega\right\}. \qquad (27)$$

In the case (25)

$$A_\sim = \frac{e^2 \Omega^3 R_0^2 n T}{4c^3}\left\{\int\limits_0^{\arccos\{(1/\beta_0 n)(1+2/\omega_c)\}} \frac{\sin^3\theta d\theta}{(1-\beta_0 n\cos\theta)^4} - \right.$$
$$\left. - \int\limits_{\arccos\{(1-2/\omega_c)\cdot(1/\beta_0 n)\}}^{\pi} \frac{\sin^3\theta d\theta}{(1-\beta_0 n\cos\theta)^4}\right\}. \qquad (28)$$

Thus radiation propagated outside the Cerenkov cone, corresponding to the second integral in (27), (28), leads to damping of the oscillations, whereas radiation inside that cone (anomalous Doppler effect), corresponding to the first integral in (27), (28) is greater than the first one. This means that damping of the oscillator will always be observed in an isotropic medium, and $A \to 0$ only if $\beta_0 n(\omega) \to \infty$ in the essential material region of integration.

Work [14] also contains an examination of the motion of an oscillator, oscillating perpendicularly to tis progressive velocity v_0. Here, as well as in the preceeding case, damping of the oscillations is always observed in an isotropic medium.

In order to explain certain peculiarities of charges moving at speeds faster than that of light in an isotropic media, it is convenient to examine the motion of an oscillator along the optical axis of a uniaxial, non-gyrotropic crystal; an electron is then considered as oscillating in the same direction. Then

$$R = (0;0;v_0 t + R_0 \sin \Omega t); \qquad k = (0; k \sin\Theta; k\cos\Theta);$$
$$e_2 = (1;0;0); \quad e_1 = (0; \cos\Theta + K_1 \sin\Theta; -\sin\Theta + K_1 \cos\Theta);$$
$$K_1 = \frac{(n_1^2 - \varepsilon_\perp)\cos\Theta}{\varepsilon_\perp \sin\Theta}; \quad \frac{1}{n_1^2} = \frac{\sin^2\Theta}{\varepsilon_\parallel} + \frac{\cos^2\Theta}{\varepsilon_\perp}; \quad kR_0 \ll 1,$$

where n_1 is the refractive index for the extraordinary wave, which is the only one to be radiated in the present case. The quantity K_1, is the ratio of the components of the electric field intensity in the extraordinary wave, paralled and perpendicular to the vector $\underline{k}$; this electric vector is parallel to the polarization vector $\underline{e}_1$, the modulus of which satisfies the condition (see [2]):

$$\varepsilon_{\parallel} a_{z_1}^2 + \varepsilon_{\perp} a_{y_1}^2 = n_1^2.$$

Then it is easy to get expressions corresponding to formulas (23) and (27):

$$A = -\frac{e^2 R_0^2 T}{4c^2 \beta_0} \int_{L_1 + L_2} \omega^2 \frac{\varepsilon_{\perp}^2(\omega) \sin^2\Theta \, d\omega}{[\varepsilon_{\perp}(\omega)\sin^2\Theta + \varepsilon_{\parallel}(\omega)\cos^2\Theta]^2 |1 - (\operatorname{ctg}\Theta / n_1)\partial n_1/\partial\Theta|}; \quad (29)$$

$$A_- = \frac{e^2 R_0^2 \Omega T}{4c^2 \beta_0} \left\{ - \int_{L_1} \omega^2 \frac{\varepsilon_{\perp}^2(\omega) \sin^2\Theta d\omega}{[\varepsilon_{\perp}(\omega)\sin^2\Theta + \varepsilon_{\parallel}(\omega)\cos^2\Theta]^2 |1 - (\operatorname{ctg}\Theta/n_1)\partial n_1/\partial\Theta|} + \right. \quad (30)$$

$$\left. + \int_{L_2} \omega^2 \frac{\varepsilon_{\perp}^2(\omega) \sin^2\Theta \, d\omega}{[\varepsilon_{\perp}(\omega)\sin^2\Theta + \varepsilon_{\parallel}(\omega)\cos^2\Theta]^2 |1 - (\operatorname{ctg}\Theta/n_1)\partial n_1/\partial\Theta|} \right\}.$$

Here the means of integrating L_1 and L_2 are determined by the Doppler relationships

$$1 - \beta_0 n(\omega\Theta)\cos\Theta = \Omega/\omega \quad (31)$$

for normal Doppler frequencies and

$$\beta_0 n(\omega\Theta)\cos\Theta - 1 = \Omega/\omega \quad (32)$$

for anomalous Doppler frequencies. It is readily seen that both integrals in (30) are always positive. This means that the radiation of normal Doppler frequencies (the first integral in (30)), corresponds to damping of the oscillations, while the radiation of anomalous Doppler frequencies corresponds to the build-up of oscillations.

It should be noted that such a division in somewhat

arbitrary, and of course, only the force, equal to the difference between the two integrals, has a physical significance.

In contrast with the isotropic case, in the problem under discussion there may occur not only an attenuation but also a build-up of oscillations. Let, for example $\varepsilon_{\parallel} < 0$, $\varepsilon_{\perp} > 0$ and let $n(\omega)$ be given by step function (25) and

$$\omega_c > \Omega\left(\sqrt{\varepsilon_{\perp}}\,\beta_0 - 1\right)^{-1}, \tag{33}$$

where $\Omega(\sqrt{\varepsilon_{\perp}}\,\beta_0 - 1)^{-1}$ is the highest possible value of the Doppler frequency ($\sqrt{\varepsilon_{\perp}}\,\beta_0 > 1$). Then, passing in (30) to integration with respect to $d\theta$, with the aid of (31) and (32) and taking into account (33), we obtain:

$$A_{\sim} = \frac{e^2\Omega^4 R_0^2 T}{4c^3}\left\{\int_0^{\operatorname{arc\,tg}\sqrt{|\varepsilon_{\parallel}|/\varepsilon_{\perp}}} \frac{n^5(\theta)\sin^3\theta\, d\theta}{\varepsilon_{\parallel}^2\left|\beta_0 n(\theta)\cos\theta - 1\right|^4} - \right. \tag{34}$$

$$\left. - \int_0^{\operatorname{arc\,tg}\sqrt{|\varepsilon_{\parallel}|/\varepsilon_{\perp}}} \frac{n^5(\theta')\sin^3\theta'\, d\theta'}{\varepsilon_{\parallel}^2\left|1 + \beta_0 n(\theta')\cos\theta'\right|^4}\right\},$$

where $\theta' = \pi - \theta$.

It is easy to see that the first integral in (34) is always greater than the second one. This means that a build-up of the oscillations takes place in this particular case.

Note that in this case there is no Cerenkov radiation, since, when $\sqrt{\varepsilon_{\perp}}\,\beta_0 > 1$, equation (1) has no real roots. The question of the motion of charges in a magnetically active plasma is dealt with in [26]. There it is shown that under certain conditions there may occur a build-up of oscillations.

The following concrete cases are examined in detail

a)	$\beta_0 = 0{,}01$,	$\omega_0^2/\omega_H^2 = 0{,}01$;
b)	$\beta_0 = 0{,}01$,	$\omega_0^2/\omega_H^2 = 10$;
c)	$\beta_0 = 0{,}99$,	$\omega_0^2/\omega_H^2 = 0{,}01$;
d)	$\beta_0 = 0{,}99$,	$\omega_0^2/\omega_H^2 = 10$.

where $\omega_0^2 = \frac{4\pi e^2 N}{m}$; $\omega_H = \frac{e H_0}{mc}$; H_0 is the homogeneous magnetic field, in which the plasma is placed, N is the electron density of the plasma. It proved that in cases a, b and d a build-up of oscillations takes place, whereas damping is encountered in case c . Differences in the sign of the forces acting on the oscillating motion of a particle in normal and anomalous Doppler radiation are obviously in full agreement with the conclusions reached in section 1. In the isotropic case this leads to a weakening in the "friction" or even to its virtual disappearance but it cannot cause a build-up of oscillations (the quantum effect of the latter, connected with spreading of the packet in the "energy space", is observed, of course, in the case of radiation at speeds faster than light in an isotropic medium also (see section 1.).) In an anisotropic medium, in particular in a magnetoactive plasma, a build-up of oscillations is possible.

Clearly, the instability of beams of particles at speeds faster than light, found in a classical approximation even in an isotropic medium, is related to the radiation reaction examined above for a single particle.

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ON SCATTERING OF RADIO-WAVES IN THE IONOSPHERE

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The question of the scattering of radio-waves in a plane-stratified medium with a heterogeneous dielectric constant is examined. An expression has been obtained for scattered energy in the case of the normal incidence of a plane wave. The problem has been solved for the case of small angles of scattering for the oblique incidence of a plane wave and the normal incidence of a spherical wave. The results are used to estimate fluctuations of electron density in the ionosphere.

Fluctuations in the amplitudes and angles of incidence of radio waves reflected from the ionosphere, the flicker of "radio stars", and also the long-range propagation of short radio waves are due to the presence in the ionosphere of inhomogeneities in electron density, which are partly characterized by the mean values of fluctuations in electron density and by the dimensions of the ionospheric inhomogeneities. In this context there arises the question of the scattering of radio waves at inhomogeneities in the dielectric constant ε, the mean value of which changes slowly with altitude. As it applies to the ionosphere, this question has been tackled by Al'pert [1] and Denisov [2] by the method of small perturbations. In deriving the effective scattering diameter in [1], it was considered that the dielectric constant, when averaged along the whole path, is close to unity; however, the altitude-dependence of $(\overline{\Delta\varepsilon})^2$ was taken into

account. A stricter examination of the case of small angles of scattering and normal incidence of waves on the layer has been carried out in [2]. There the refraction of scattered waves was taken into account in the approximation of geometric optics, and the effective cross-section of scattering was calculated by averaging the energies of scattered waves after their egress from ionosphere.

A general solution for the normal incidence of a plane wave and an approximate one (for small angles of scatter) for oblique incidence have been obtained in the present work. An approximate solution for the normal incidence of a spherical wave has also been obtained. In calculating scattered energy we have utilized the effective cross section of scatter for a homogeneous medium, with a subsequent allowance for refraction of scattered waves, which, in the approximation of geometric optics, is equivalent to the method used in [2]. The results thus obtained were used to estimate the values of electron concentration fluctuations in the ionosphere.

1. Let us review the general expression for scattered energy at the point of observation. Assume a medium with random deviations of the dielectric constant ε from its mean value $\bar{\varepsilon}$, which depends only on one coordinate z. Assume also that the characteristic scale of random inhomogeneities is far smaller than the scale of regular changes in $\bar{\varepsilon}$, and that random deviations $\Delta\varepsilon \ll \bar{\varepsilon}$. In calculating scattered energy, it is convenient to use effective cross section of scatter σ, determined by the relationship:

$$P_s = p_0(\varepsilon, \vartheta, \psi)\, dV\, d\Omega, \tag{1}$$

where p_s is the energy flux of scattered waves in the solid angle $d\Omega$ in a direction which forms angle ϑ with the direction of the incident wave, p is the density of the energy flux of the incident wave, dV is an element of volume and ψ is the angle between the vector of the electric field of the incident wave and the direction of scatter.

In order to simplify the calculations we shall assume the absence of absorption.

We shall assume that the wave falls on the layer at an angle ϑ_{01} to the axis of z (see Fig. 1)*. We shall denote by ϑ_1 the angle between the direction of scatter of the incident wave and the z-axis at the point of scatter, and by ϑ_2 the angle between the z-axis and the direction of propagation of the scattered wave. In this case the angle of scatter $\vartheta = \vartheta_1 + \vartheta_2$ and $\sqrt{\varepsilon_1} \sin \vartheta_1 = \sqrt{\varepsilon_0} \sin \vartheta_{01}$, where ε_1 and ε_0 are the values of the dielectric constant at the point of scatter and inside the layer. Let us single out an area ds_2 at the point of observation, which is perpendicular to the z-axis. The density of the flux of scattered energy through this area from the scattering volume dV will, in conformity with (1), be

$$p = dV \, d\Omega_2 / ds_2,$$

where $d\Omega_2$ is the solid angle formed by a cone of rays based on the area ds_2. Consequently, the total density of the flux

* Fig. 1 refers to the case of scattering of a spherical wave, which will be examined below. In the case of a plane wave, it should be considered that the solid angle $d\Omega_{01} = 0$.

of scattered energy may be written in the form

$$P_s = \int\int\int p\sigma \frac{d\Omega_2}{ds_2} dV, \qquad (2)$$

where integration is carried out with respect to the entire scattering volume. We shall consider only a medium in which the region of existence of inhomogeneities is limited vertically by the planes $z = z_1$ and $z = z_2$. Since $d\Omega_2 = \sin \vartheta_2 d\vartheta_2 d\varphi$ where φ is the azimuth angle, then, assuming that in (2) $dV = ds_2 dz$, we have:

$$P_s = \int_0^{2\pi} \int_0^{\pi/2} \int_{z_1}^{z_2} p\sigma \sin \vartheta_2 d\varphi d\vartheta_2 dz, \qquad (3)$$

where p and σ in general depend on the coordinates φ, ϑ_2 and z.

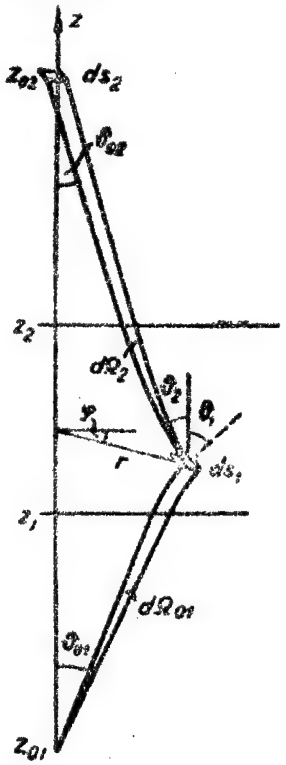

Fig. 1

If the region in which inhomogeneities exist is limited also in the horizontal direction, or if it is necessary to take into account the directional diagrams of transceiver systems, as happens in the case of long-range propagation of waves due to scattering, then formula (3) will become somewhat more complicated. We shall not be concerned with these questions in the present chapter.

Before passing on to the calculation of integral (3), we shall examine certain energy relationships obtaining in the scattering of a plane wave. Let the wave fall on the layer at an angle ϑ_{01}. Then, without including scattering, density of energy flux at any point $p = p_0 \cos\vartheta_{01}/\cos\vartheta_1$, where $\cos\vartheta_1 = \sqrt{1 - \varepsilon_0 \sin^2\vartheta_{01}/\varepsilon_1}$, and p_0 is the density of energy flux of the incident wave. If we can neglect energy scattered into the half-space $z < z_1$ whence the wave is falling, then in the path dz the density of the energy flux of the incident wave P along the z-axis varies according to the magnitude of the scattered energy, i.e.

$$-dP = dP = P \cos^{-1}\theta_1 f(z)\, dz,$$

where in accordance with (3)

$$f(z) = \int_0^{2\pi}\int_0^{\pi/2} \sigma \sin\theta_2\, d\varphi\, d\theta_2. \qquad (4)$$

Hence we have

$$P(z) = P_0 \exp\left[-\int_{z_1}^{z} f(z) \cos^{-1}\theta_1\, dz\right],$$

where $$P_0 = p_0 \cos\theta_{01} = \text{const}.$$

In accordance with the law of conservation of energy, after egress from the layer $P_0 = P_s + P$ and the ratio of scattered energy P_s to unscattered energy P will be equal to:

$$P_s/P = \exp\left[\int_{z_1}^{z_2} f(z)\cos^{-1}\vartheta_1\, dz\right] - 1. \qquad (5)$$

If scattering is small, i.e. $P_s \ll P$, then expanding the exponent in series, we obtain:

$$P_s/P = \int_{z_1}^{z_2} f(z)\cos^{-1}\vartheta_1\, dz. \qquad (6)$$

Since $dz/\cos\vartheta_1 = dl$ is an element of the propagation path of the wave, integration in (5) will naturally be carried out along this path. Note that the calculation of scattered energy from each segment of integration of dl is made as for a homogenous medium, while the scattered energy does not depend on the properties of the medium along the path from the scattering volume to the point of observation.

Further, in calculating the function f (z), it is necessary to concretize the dependence of σ on the coordinates. The form of the $f(\sigma)$ depends essentially on the selection of the correlation function of the random quantity $\Delta\varepsilon$. As has been remarked in [1], in the case of the ionosphere it is most appropriate to use a correlation function of the gaussian type

$$\rho(r) = \exp(-r^2/l^2),$$

where ξ is the characteristic dimension of the inhomogeneities*. Using the known method of finding σ (see, for example, [1]), it can be shown that in a medium with dielectric constant ε the effective diameter of cross section is

$$\sigma(\varepsilon, \vartheta, \psi) = (\overline{\Delta\varepsilon})^2 \frac{\sqrt{\pi}}{8\lambda_0} a^3 \sin^2\psi \exp\left\{-\varepsilon\left(a \sin\frac{\vartheta}{2}\right)^2\right\}, \qquad (7)$$

where $a = 2\pi\xi/\lambda_0$ and λ_0 is the wavelength in a vacuum. It is necessary that the point of observation be located at a sufficient distance from the scattering volume so that $\sqrt{\lambda_0 R} \gg \xi$, where R is the group path of the scattered wave.

In the case of normal incidence of a plane wave the angle of scattering $\vartheta = \vartheta_2$ and $\sin^2\psi = 1 - \sin^2\vartheta_2 \sin^2\varphi$. Substituting these values in (7), and (7) in (4), we have:

$$f(z) = \frac{\sqrt{\pi}}{8\lambda_0} a^3 (\overline{\Delta\varepsilon})^2 \int_0^{2\pi}\int_0^{\pi/2} \exp\left\{-\varepsilon\left(a \sin\frac{\vartheta}{2}\right)^2\right\} \times$$

$$\times (1 - \sin^2\vartheta_2 \sin^2\varphi) \sin\vartheta_2 \, d\varphi \, d\vartheta_2,$$

whence, having carried out the integration, we find:

$$f(z) = \pi^{3/2} \frac{\xi}{\lambda_0^2} \frac{(\overline{\Delta\varepsilon})^2}{\varepsilon}\left[1 - \frac{2}{a^2\varepsilon} + \frac{4}{a^4\varepsilon^2} - \left(\frac{1}{2} + \frac{4}{a^4\varepsilon^2}\right)\exp\left(-\frac{a^2\varepsilon}{2}\right)\right]. \qquad (8)$$

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* In his diploma thesis B.V. Bezrukov (Gor.'kiy University, 1959) makes a similar derivation for a correlation function of the type

$$\rho(r) = e^{-r/\xi} \quad \text{и} \quad \rho(r) = \frac{1}{[1 + (r/\xi)^2]^2}.$$

In the case of small angles of scattering the final results coincide accurately except for a multiplier of the order of unity.

Substitution of this relationship in (5), taking into account the fact that $\cos \varphi_1 = 1$, gives a general expression for the ratio of the fluxes of the scattered and the unscattered waves at the point of observation for normal indicende of a plane wave on the layer:

$$\frac{P_s}{P} = \exp\left\{\frac{\pi^{5/2}}{\lambda_0^2}\int_{z_1}^{z_2}\xi\frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon}\left[1-\frac{2}{a^2\varepsilon}+\frac{4}{a^4\varepsilon^2}-\left(\frac{1}{2}+\frac{4}{a^4\varepsilon^2}\right)\times\right.\right. \tag{9}$$

$$\left.\left.\times\exp\left(-\frac{a^2\varepsilon}{2}\right)\right]dz\right\}-1.$$

It should be noted that the ratio Ps/P has no singularity when $\varepsilon \to 0$ or when integrating with respect to the finite part of the variation in ε. However, it should be remembered that, generally speaking, when $\varepsilon \to 0$ geometric optics become inapplicable and the condition that the disturbance in the medium be small ($\Delta\varepsilon \ll \varepsilon$) is not fulfilled.

Expressions (8) and (9) are substantially simplified in two cases. If

$$a^2\varepsilon = (2\pi\xi/\lambda_0)^2\varepsilon \gg 1$$

(which corresponds to scattering at large inhomogeneities) then

$$f(z) = \frac{\pi^{5/2}}{\lambda_0^2}\xi\frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon}$$

and, consequently

$$\frac{P_s}{P} = \exp\left(\frac{\pi^{5/2}}{\lambda_0^2}\int_{z_1}^{z_2}\xi\frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon}dz\right)-1. \tag{10}$$

If $P_s \ll P$ (scattered energy small), then

$$\frac{P_s}{P} = \frac{\pi^{5/2}}{\lambda_0^2}\int_{z_1}^{z_2}\xi\frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon}dz, \tag{11}$$

Which coincides with results obtained in work [2].

In the case of scattering at small inhomogeneities, when $a^2 \varepsilon \ll 1$,

$$f(z) = \frac{4}{3} \frac{\pi^{9/2}}{\lambda_0^4} \xi^3 \overline{(\Delta\varepsilon)^2},$$

and if $P_s \ll P$, then as would be expected, we obtained a Rayleigh law of scattering

$$\frac{P_s}{P} = \frac{4}{3} \frac{\pi^{9/2}}{\lambda_0^4} \int_{z_1}^{z_2} \xi^3 \overline{(\Delta\varepsilon)^2}\, dz. \qquad (12)$$

Let us calculate the expression for P_s/P for the case of oblique incidence of a plane wave. In this case

$$\sin^2\psi = 1 - \sin^2\vartheta_2 \sin^2\varphi';$$

$$\cos\vartheta = \sin\vartheta_1 \sin\vartheta_2 \cos\varphi + \cos\vartheta_1 \cos\vartheta_2,$$

where φ' is the azimuth angle in the plane perpendicular to the direction of propagation of the incident wave. Let the angle of scattering be small ($\vartheta \ll 1$), and the angle of inclination of the trajectory of the ray to z-axis significantly different from zero. Under these conditions it can be considered that

$$\sin^2\psi \simeq 1; \quad \cos\varphi = 1 - \varphi^2/2;$$

$$1 - \cos\vartheta = 1 - \cos(\vartheta_2 - \vartheta_1) + (\varphi^2/2)\sin\vartheta_1 \sin\vartheta_2 \simeq$$

$$\simeq (\vartheta_2 - \vartheta_1)^2/2 + (\varphi^2/2)\sin\vartheta_1 \sin\vartheta_2.$$

Substituting these relationships in (7), and (7) in (4), we obtain:

$$f(z) = \frac{\sqrt{\pi}\, a^3}{8\lambda_0} \int_0^{2\pi} \int_0^{\pi/2} \overline{(\Delta\varepsilon)^2} \exp\left\{ -\frac{\varepsilon a^2}{4} (\vartheta_2 - \vartheta_1)^2 - \right.$$

$$\left. - \frac{\varepsilon a^2}{4} \sin\vartheta_1 \sin\vartheta_2 \varphi^2 \right\} \sin\vartheta_2\, d\varphi\, d\vartheta_2. \qquad (13)$$

Under the integral there are functions, the values of which rapidly diminish with change in the variables φ and $\vartheta_2 - \vartheta_1$. Therefore, on integrating these variables from 0 to a certain sufficiently high value, one can exchange the upper limit of integration for infinity. Proceeding in the above manner, we have:

$$\int_0^{2\pi} \exp\left(-\frac{\varepsilon a^2}{4}\sin\vartheta_1 \sin\vartheta_2 \varphi^2\right) d\varphi = \frac{2}{a}\sqrt{\frac{\pi}{\varepsilon \sin\vartheta_1 \sin\vartheta_2}}. \quad (14)$$

In the integral obtained after substituting (14) in (13)

$$\int_0^{\pi/2} \exp\left\{-\frac{\varepsilon a^2}{4}(\vartheta_2 - \vartheta_1)^2\right\} \sqrt{\frac{\sin\vartheta_2}{\sin\vartheta_1}}\, d\vartheta_2 \quad (15)$$

we have the product of rapidly diminishing function and the slowly changing function $\sqrt{\sin\vartheta_2/\sin\vartheta_1}$. Exchanging the variable ϑ_2 for $\vartheta_2 - \vartheta_1 = \beta$ and considering that when $\vartheta_2 = \vartheta_1$, $\sqrt{\sin\vartheta_2/\sin\vartheta_1} = 1$, we find instead of (15):

$$2\int_0^{\infty} \exp\left\{-\frac{\varepsilon a^2}{4}\beta^2\right\} d\beta = \frac{2\sqrt{\pi}}{a\sqrt{\varepsilon}}. \quad (16)$$

Substituting the values obtained in (13, and (13) in (5), we finally find that

$$\frac{P_f}{P} = \exp\left(\frac{\pi^{5/2}}{\lambda_0^2}\int \varepsilon \frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon} dl\right) - 1, \quad (17)$$

where integration is carried out along the path of wave propagation.

The above derivation was made for an angle ϑ_1, significantly different from zero. On the other hand, at $\vartheta_1 \to 0$, ratio (17) passes at the limit into expression (10), which is true for $\vartheta_1 = 0$. For this reason it can, evident-

ly, be considered that formula (17) is true for any given ν_1^{ℓ}.

Let us pass to the scattering of spherical waves. We shall limit ourselves to the particular case where the radiating point and the receiving point are on the z-axis (Fig. 1). At the point of scattering we shall choose an area ds_1, perpendicular to the direction of propagation of the incident wave, and by $d\Omega_{01}$ we shall designate the solid angle at the point of radiation, formed by a cone of rays based on the area ds_1. Then, without taking into account the loss of energy in scattering, the density of energy flux at the point of scattering is

$$p = p_{\nu} d\Omega_{01}/ds_1, \tag{18}$$

where p_{ν} is the flux of energy from the source per unit of solid angle. Substituting (18) in (3), we have:

$$P_2 = \int_0^{2\pi}\int_0^{\pi/2}\int_{z_1}^{z_2} \sigma\,(d\Omega_{01}/ds_1)\sin\vartheta_2\, d\varphi\, d\vartheta_2\, dz\,. \tag{19}$$

Let r designate the distance from the point of scatter to z-axis. Then

$$\frac{d\Omega_{01}}{ds_1} = \frac{\sin\vartheta_{01}\, d\vartheta_{01}\, d\varphi}{r\,(\partial r/\partial\vartheta_{01})\, d\vartheta_{01}\, d\varphi}\,\frac{1}{\cos\vartheta_1} = \left[\varepsilon_0 R_1 \frac{\partial}{\partial\vartheta_{01}}(R_1\sin\vartheta_{01})\cos\vartheta_1\right]^{-1},$$

where ε_0 is the value of ε at the point where the primary emitter is located, and

$$R_1 = \int_{z_m}^{z}\frac{dz}{\sqrt{\varepsilon\,(1-\sin^2\vartheta_1)}} = \frac{r}{\sqrt{\varepsilon}\sin\vartheta_1}$$

is the group path from the source to the scattering volume. Substituting the reduced expressions in (19) we find:

$$P_s = p_s \int_0^{2\pi} \int_0^{\pi/2} \int_{z_1}^{z_2} \frac{\varepsilon \sin\vartheta_2 \, d\varphi \, d\vartheta_2 \, dz}{\varepsilon_0 R_1 \cos\vartheta_1 \, \partial (R_1 \sin\vartheta_{01})/\partial\vartheta_{01}}. \tag{20}$$

We observe that

$$R_1 \sin\vartheta_1 = R_2 \sin\vartheta_2, \tag{21}$$

where

$$R_2 = \int_z^{z_2} \frac{dz}{\sqrt{\varepsilon(1-\sin^2\vartheta_2)}} = \frac{r}{\sqrt{\varepsilon}\sin\vartheta_2}$$

is the group path from the point of scattering to the point of observation. Further, let us limit ourselves to the case of small angles of scatter, when $\vartheta = \vartheta_1 + \vartheta_2 \ll 1$. in this case

$$\begin{gathered} R_1\vartheta_1 \simeq R_2\vartheta_2; \quad \sin^2\psi \simeq 1; \quad \partial(R\sin\vartheta_{01})/\partial\vartheta_{01} \simeq R_1; \\ \sin\frac{\vartheta}{2} = \sin\frac{\vartheta_1+\vartheta_2}{2} \simeq \frac{R_2+R_1}{2R_1}\vartheta_2. \end{gathered} \tag{22}$$

Substituting these expressions in (20), and taking into consideration (7), we obtain:

$$P_s = p_s \frac{\sqrt{\varepsilon}\, a^2}{8\vartheta_0^2 \varepsilon_0} \int_0^{2\pi} \int_0^{\pi/2} \int_{z_1}^{z_2} \frac{(\Delta\varepsilon)^2}{R_1^2} \exp\left\{-a^2\varepsilon\left(\frac{R_2+R_1}{2R_1}\right)^2\vartheta_2^2\right\}\vartheta_2 \, d\varphi \, d\vartheta_2 \, dz.$$

Since at small angles the values R_1 and R_2 are weakly dependent on ϑ_2, then by replacing R_1 and R_2 with their values at $\vartheta_2 = 0$, we obtain after integrating with respect to φ and ϑ_2,

$$P_s = \frac{P_u}{(\overline{R_1} + \overline{R_2})^2} \frac{\pi^{5/2}}{\varepsilon_0 \lambda_0^2} \int_{z_1}^{z_2} \xi \frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon} dz. \qquad (23)$$

The ratio $P_u/(\overline{R_1} - \overline{R_2})^2$ in our case is the density of the energy flux of the undisturbed wave at the point of observation. Consequently, exchanging this ratio for P, we obtain:

$$\frac{P_s}{P} = \frac{\pi^{5/2}}{\lambda_0^2} \int_{z_1}^{z_2} \xi \frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon} dz. \qquad (24)$$

Comparing this formula with expression (11) obtained for the plane wave, we see that in the case of scatter at large inhomogeneities, the ratio of scattered and unscattered energy is expressed in the same way for both the plane and the spherical waves.

A certain analogy can be continued further. At small angles of scatter one can, as a first approximation, consider that at the point of observation the decrease in the density of the energy flux of the basic wave, caused by scatter, is numerically equal to the density of the flux of scattered energy. Then, by analogy with the case of the plane wave, it can be shown that

$$\frac{P_s}{P} = \exp\left(\frac{\pi^{5/2}}{\lambda_0^2} \int_{z_1}^{z_2} \xi \frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon} dz \right) - 1. \qquad (25)$$

However, it must be noted that the angular spectrum of the scattered wave, as is seen from (22), differs from the case of plane wave, where $\mathcal{D}_1 = \mathcal{D}_2$.

The formulas obtained show that the contribution by

different regions to the energy of scattered waves depends on the value of $\xi(\Delta\varepsilon)^2/\varepsilon$ for each region. Apparently in the ionosphere all three values: ξ, $(\Delta\varepsilon)^2$ and ε vary with altitude; however, only the change in ε is reliably known. Therefore, in concrete calculations certain supplementary assumptions have to be made with respect to the change in ξ and $(\Delta\varepsilon)^2$ with altitude.

2. Let us utilize results obtained so far to calculate electron density fluctuations in the ionosphere. $(\Delta N/N)^2$. The method of determining $(\Delta N/N)^2$ from the fading of a signal reflected from the ionosphere was proposed by Al'pert [1]. Numerical evaluations, obtained on the basis of this method, are given in references [1, 4].

Let us examine the effect of the change in ε with altitude on the value of $(\Delta N/N)^2$, calculated from experimental data.

It was shown above that in the case of normal incidence of a spherical wave on a scattering layer, the ratio of scattered to undisturbed energy at the receiving point is

$$\frac{P_s}{P} = \frac{\pi^{5/2}}{\lambda_0^2}\int_{z_1}^{z_2} \xi\frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon}\,dz.$$

In the case of an ionized gas

$$\varepsilon = 1 - 4\pi e^2 N/m\omega^2,$$

where e and m are the charge and mass of an electron, $\omega = 2\pi c/\lambda$, N is the electron density. Hence

$$\overline{(\Delta\varepsilon)^2} = (1-\varepsilon)^2\,\overline{(\Delta N/N)^2}. \qquad (26)$$

If it is assumed, as in [1], that the values of $\overline{(\Delta N/N)^2}$

and ξ change little with altitude, then expression (24) can be rewritten in the following way:

$$\frac{P_s}{P} = \left(\frac{\Delta N}{N}\right)^2 \pi^{3/2} \frac{\xi}{\lambda_0^2} \int_{z_1}^{z_2} \frac{(1-\varepsilon)^2}{\varepsilon} dz. \qquad (27)$$

Relationships (24) and (27) are true under the following conditions: 1) the angles of scatter are small, i.e. $\vartheta^2 \ll 1$, or for angles after egress from ionosphere $z_0^2 \ll \varepsilon$; 2) the approximation of geometric optics is fulfilled $(\lambda_0/2\pi)\left|(d\varepsilon/dz)\,\varepsilon^{-3/2}\right| \ll 1$; 3) fluctuations in the dielectric constant are small, i.e. $\Delta\varepsilon \ll \varepsilon$; 4) the point of observation is located sufficiently far from the scattering volume, so that $\sqrt{\lambda_0 R} \gg \xi$, where R is the group path of the scattered waves; 5) the scattered energy is small: $P_s \ll P$.

Let us examine a parabolic model of the layer

$$\varepsilon = 1 - b^2\left(1 - z^2/z_m^2\right),$$

where $b = \lambda_0/\lambda_c$, λ_c is the critical wavelength, z_m is the half-thickness of the layer and z is read from the maximum of the layer. In order to evaluate the ratio $(\Delta N/N)^2$, let us consider a numerical example, using some typical values of parameters of the layer F: $\lambda_0 = 62.5$ m, $\lambda_c = 50$ m, $z_m = 100$ km, the altitude of the beginning of the layer $z_0 = 200$ km, dimensions of inhomogeneities $\xi = 200$ m, and the ratio $P_s/P = 0.1$. At these parameter values conditions 4) and 5) are fulfilled throughout, and conditions 1) - 3) are violated only near the point of reflection, where is sufficiently small. It is known [1] from experiment that the angle of scatter after egress from the layer does not exceed $2 \div 3^\circ$, hence it is necessary for the fulfillment of condition 1) that $\varepsilon \gg 3 \cdot 10^{-3}$. For the selected model condition 2) also

gives $\varepsilon \gg 10^{-3}$. Fluctuations in the dielectric constant, when ε is close the zero, are numerically equal to the corresponding fluctuations in electron density (26), and condition 3) can be rewritten in the form: $\varepsilon \gg \sqrt{(\overline{\Delta N/N})^2}$. Thus, if the value of $\sqrt{(\overline{\Delta N/N})^2} < 10^{-3}$, then expression (27) is true up to the level $z - z'$, where $\varepsilon' = 10^{-2}$. For the total scattered energy, taking into account the entire path of wave to the point of reflection and back, we can consequently write:

$$\frac{P_s}{P} = \left(\overline{\frac{\Delta N}{N}}\right)^2 \left(2\pi^{5/2} \frac{\xi}{\lambda_0^2} \int_{z_1}^{z_*} \frac{(1-\varepsilon)^2}{\varepsilon}\, dz + J\right). \qquad (28)$$

where J takes into account the contribution of the region $\varepsilon < 10^{-2}$ to the scattered energy. If we limit ourselves to the first componant discarding J, we shall still obtain only the upper limit of $(\overline{\Delta N/N})^2$, i.e.

$$\left(\overline{\frac{\Delta N}{N}}\right)^2 \leqslant \frac{P_s}{P}\left(2\pi^{5/2} \frac{\xi}{\lambda_0^2} \int_{z_1}^{z'} \frac{(1-\varepsilon)^2}{\varepsilon}\, dz\right)^{-1} = \frac{P_s}{P}\left(2\pi^{5/2} \frac{\xi}{\lambda_0^2} I\right)^{-1}. \qquad (29)$$

For a parabolic layer the integral

$$I = \frac{z_m}{b}\left[\frac{1}{b^2-1} \ln\left(\frac{1}{\sqrt{\varepsilon'}} \frac{\sqrt{b^2-1} + \sqrt{b^2-1+\varepsilon'}}{\sqrt{b^2-1}+b}\right) - \right.$$
$$\left. - (1+b^2)\left(b - \sqrt{b^2-1+\varepsilon'}\right) + \frac{1}{3}\left(b^3 - \sqrt{(b^2-1+\varepsilon')^3}\right)\right], \qquad (30)$$

where $b = \lambda_0 / \lambda_c$.

Substituting the above numerical values for the parameters of the ionospheric layer, we obtain $\sqrt{(\Delta N/N)^2} \leq 4 \times 10^{-4}$, whereas for the same starting data in [1] $\sqrt{(\Delta N/N)^2} = 7 \times 10^{-3}$. Similar large differences are also obtained for other values of ionospheric parameters, which testifies to the fact that the allowance for change in with altitude is generally speaking right. Thus, for example, for a layer E with $\lambda_0 = 94$ m, $\lambda_c = 75$ m, $z_m = 200$ km, $\xi = 200$ m, it follows from conditions 1) - 3) that the upper limit of integration $\varepsilon' = 5 \times 10^{-2}$. then from (29) - (30) we obtain: $\sqrt{(\Delta N/N)^2} \leq 2 \times 10^{-3}$, whereas according to [1] $\sqrt{(\Delta N/N)^2} = 10^{-2}$. As is seen from (27), scattering increases with a decrease in ε, other conditions being equal.

Calculations according to formulas (29) and (30), show, that for the chosen model of layer F, one half of the total amount of scattered energy comes from the region of thickness $\Delta \approx 4$ km, situated below the level z'. The value obtained for $(\Delta N/N)^2$ refers in fact to a small region of altitudes near the point of reflection. The question of the effect on the value of the scattered energy of the region where $\varepsilon \sim 0$ is not sufficiently clear. Nevertheless, we may note that at frequencies not very near to the critical, expression (30) depends little on ε', since ε' comes under the logarithm, and a three or four fold change in ε' alters the expression $\sqrt{(\Delta N/N)^2}$ by only 10 - 20%. Furthermore, to the upper limit of integration $\varepsilon' = 10^{-2}$ there corresponds the point z', which is located only 1 km below the point where $\varepsilon = 0$, which, in accordance with [3], attain values of the order of $15 \lambda_0 \sim 1$ km. Therefore it can be

assumed that the energy scattered by this region is relatively small. If we apply formally Rayleigh's formula for scattering (12), then along the path $\Delta z = 1000$ m, $\xi = 200$ m, $\lambda_0 = 62.5$ m and $(\overline{\Delta\varepsilon})^2 = (\overline{\Delta N/N})^2 \cong 1.6 \times 10^{-7}$, we get the ratio $P_s/P \cong 10^{-2}$, i.e. the contribution of this region to the scattering energy is small.

The same conclusion can be reached from an examination of the question of the influence of the Earth's magnetic field on the scattering of radio waves in the ionosphere. As a rule, experiments aim at measuring fluctuations of one of the magneto - ionic components of a signal reflected from ionosphere. It is also known [5] that on the average the depth of fluctuations of an extraordinary signal is greater than of an ordinary one. For instance, the ratio for layer F

$$(P_s/P)_1/(P_s/P)_2 \simeq 1{,}7,$$

where indices 1 and 2 correspond to the ordinary and extraordinary components. A similar effect is connected with the fact that the refractive indices of the ordinary and extraordinary components $n_{1,2}$ depend in different ways on the altitude. If the effects of the interaction of radio and waves inhomogeneities and of the anisotropy of the dimensions of the inhomogeneities are not taken into account, then it is easy to show that in quasilongitudinal approximation formula (29) is true when ε is replaced with $n^2_{1,2} = 1 - b^2_{1,2}\left(1 - z^2/z_m^2\right)$, where

$$b^2_{1,2} = \frac{\lambda_0^2}{\lambda_c^2} \frac{1}{1 \mp (\lambda_0/\lambda_H)\cos\alpha},$$

where λ_H is the gyromagnetic wavelength and α is the angle between the Earth's magnetic field and the vertical. The

limits of applicability of quasilongitudinal approximation for the layer F of the ionosphere [3], when the angle between the Earth's magnetic field and the vertical is 20°, make it possible to use, in the case of an extraordinary wave, the level z_1' (up to which formula (29) holds true), corresponding to a value of $n_1^2 = 10^{-2}$. In the case of an ordinary wave the value of n_2^2, for which (29) holds true, is of the order of 0.1. Calculation in accordance with formula (29) for the ordinary and extraordinary components gives the ratio

$$\left(\frac{P_2}{P}\right)_1 \Big/ \left(\frac{P_2}{P}\right)_2 \approx 2{,}3.$$

If we study the contribution to the scattering of an ordinary signal, of the region $n_2^2 < 0.1$ where geometric optics are still applicable, then the above ratio would diminish, and in all probability would approach the experimental value of 1.7. This, in our opinion further confirms the conclusion that the contribution of the region of reflection to the scattered energy is not decisive.

It is also of interest to evaluate the possibility of determining $(\overline{\Delta N/N})^2$ from radioastronomical data on the flickering of discrete emitters of radio radiation. As a rule flickering is observed on waves in the meter range, where the influence of the Earth's magnetic field can be neglected and where there are no difficulties connected with the transgression of the limits of applicability of geometric optics. However, because the wavelengths are short ($\lambda_0 \sim 3$ m), and the dimensions of the inhomogeneities are large ($\xi \sim 3$ km), the conditions $\sqrt{\lambda_0 R} \gg \xi$ is violated. As shown in [6],

the scattered energy increases with an increase in the ratio $\sqrt{\lambda_0 R}/\xi$ and attains saturation at $\sqrt{\lambda_0 R}/\xi \geqslant 3$. In our case, in which $\lambda_0 = 3$ m, R = 500 km and $\xi = 3$ km, the ratio $\sqrt{\lambda_0 R}/\xi = 0.4$, and the scattered energy calculated according to formula (24) is known to be larger than that obtainable in reality. Thus,

$$\frac{P_s}{P} < \frac{\pi^{5/2}}{\lambda_0^2} \int_{z_1}^{z_2} \xi \frac{\overline{(\Delta\varepsilon)^2}}{\varepsilon} dz . \qquad (24')$$

If the values of $\overline{(\Delta N/N)^2}$ and ξ vary little with altitude, then

$$\overline{\left(\frac{\Delta N}{N}\right)^2} > \frac{P_s}{P} \left(\pi^{5/2} \frac{\xi}{\lambda_0^2} \int_{z_1}^{z_2} \frac{(1-\varepsilon)^2}{\varepsilon} dz \right)^{-1} . \qquad (29')$$

For a parabolic layer, on condition that $\lambda_0^2 \ll \lambda_c^2$,

$$\sqrt{\overline{\left(\frac{\Delta N}{N}\right)^2}} > 0{,}65 \frac{\lambda_0^2}{\lambda_0 \sqrt{\xi z_m}} \sqrt{\frac{P_s}{P}} .$$

Assuming that $\lambda_c = 60$ m, $\lambda_0 = 3$ m, $\xi = 3$ km, $z_m = 300$ km and $P_s/P = 0.1$, we obtain $\sqrt{\overline{(\Delta N/N)^2}} > 2.5 \times 10^{-3}$.

It must be mentioned that if inhomogeneities ar not spread throughout the thickness of the layer, then the ratio $\overline{(\Delta N/N)^2}$ will increase. Thus the respective fluctuations in electron density at large - scale inhomogeneities ($\xi \sim 3$ km) are larger by at least one order than in small - scale ones ($\xi \sim 200$ m). Therefore, the radioastronomical method does not record small - scale inhomogeneities, since the scattered energy is proportional to the dimensions of the

inhomogeneities and to the fluctuations in electron density. On the other hand, in the method of pulse sounding, large - scale inhomogeneities cause a slow focusing and defocusing of the signal, which leads to the disturbance of the stationary nature of the process of rapid fading occasioned by small - scale inhomogeneities.

In conclusion, the authors wish to take the opportunity of thanking B.L. Ginzburg and G.G. Getmantsev for their interest in the present work and for a number of valuable observation .

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THE INFLUENCE OF AN ELECTRIC FIELD UPON VELOCITY DISTRIBUTION OF ELECTRONS IN MOLECULAR PLASMA (IONOSPHERE)

Pages 355-369

By A.V. Gurevich

The form of the velocity distribution function for electrons in a plasma in molecular gases (in particular in the ionosphere) is found for low electron energies (up to 1 ev). Expressions are obtained for the mean electron energy $\mathcal{E}$ and for the electron current j. A comparison is made between the results of calculating $\mathcal{E}$ and j from formulas of the exact kinetic theory and results of the commonly used approximate "elementary" theory. The conditions of applicability of the elementary theory are also obtained.

Introduction

A decisive influence on the velocity distribution of plasma electrons is exerted by collisions between electrons and elastic and non-elastic collisions between electrons and heavy particles such as molecules and ions.

A majority of the theoretical works devoted to the subject neglect the influence of non-elastic collisions. This is valid for a fully ionized gas and for monatomic (noble) gases at low electron energies. The role of non-elastic collisions in monatomic gases becomes significant at higher electron energies (excitation of optical levels and ionization). The influence of such collisions on the velocity distribution of electrons has been investigated by Druyvestein, Smith, Etc. (see, for instance [1], ch. 8).

In the case of molecular gases (such as hydrogen, oxygen, nitrogen) non-elastic collisions play a decisive part even at energies of the order of 0.01 ev, i.e. at room temperatures. At the same time, very little is known at present [2] about the cross sections of non-elastic collisions between slow electrons and molecules; therefore, the problem of the velocity distribution of electrons in a molecular plasma has not been solved either. The mean energy of the electrons and the current in a molecular plasma are usually calculated by means of the simplified kinetic theory, called the "elementary" theory (see, for instance, [3], part 2): it neglects entirely the spread in electron velocities in the plasma. Therefore the values of the averages, calculated with the aid of this theory are, generally speaking, approximate. How accurate are the averages calculated with the aid of the elementary theory ? In what cases are these formulas inapplicable and what expressions should be used instead ? The answers to these questions can only be obtained from a full kinetic investigation*.

A solution of this problem is the aim of the present work. It is shown in the first section that the kinetic equation can be simplified by using the peculiarities of a molecular plasma (in analogy with the practice in the case of purely elastic collisions). The subsequent two sections give the derivation of the kinetic equation; it is analyzed for the cases of hydrogen, oxygen and nitrogen. Naturally, the experimental

* A considerable amount of research has been devoted to the investigation of molecular plasmas. The results (up to 1940) are summarized by Healy and Reed (see also [5]). Note that the ionosphere is also a molecular plasma.

data allow the determination of the function $\delta(v)$ which characterizes the role of non-elastic collisions in the final expression for the distribution function. In conclusion, the question of the accuracy of the elementary theory formulas and the limits of their applicability are discussed.

1. Basic equations.

We shall examine an unbounded plasma placed in an electric and a magnetic field. The heavy particles of the plasma are assumed to have a Maxwellian distribution. The electron distribution function f (v, r, t) is of course determined by Boltzmann's kinetic equation*.

$$\frac{\partial f}{\partial t} + v\,\mathrm{grad}_r f + \frac{e}{m}\left(E + \frac{1}{c}[vH]\right)\mathrm{grad}_v f = -S, \qquad (1)$$

where e and m are the charge and mass of an electron, c is the speed of light, E and H are the intensities of the electric and magnetic fields and S is the collision integral.

Where only elastic collisions occur, equation (1) can, as is well known, be simplified by expanding the angular part of the distribution function in the space of the velocities as spherical functions. This series converges very rapidly, since the effective parameter of the expansion is the ratio of the masses of the electron and the heavy particls m/M. Therefore it is usual to take only two terms of the expansion, i.e. to present

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* The applicability of Boltzmann's theory to a plasma is limited by the $e^2N^{1/3} \ll kT$ (potential energy of a particle should be much less than its kinetic energy) and $kT \ll (h^2/6m)(3N/\pi)^{2/3}$ (condition of non-degeneration of plasma). It is also necessary that plasma should be in a state close to equilibrium.

the distribution function in the form:

$$f(\boldsymbol{v}, \boldsymbol{r}, t) = f_0(v, \boldsymbol{r}, t) + \frac{\boldsymbol{v}}{v} f_1(v, \boldsymbol{r}, t) + \chi(\boldsymbol{v}, \boldsymbol{r}, t), \qquad (2)$$

the function χ being subsequently neglected . Eq. (1) then reduces to a system of equations for the functions f_0 and $\underline{f}_1$ [6].

It is natural to attempt to present the distribution function in the form (2) even in the presence of non-elastic collisions. Then equation (1)(if the function χ is negligible in comparison with f_0, and $\underline{f}_1^2$ in comparison with f_0^2) reduces to the following system of equations for the functions f_0 and $\underline{f}_1$:

$$\frac{\partial f_0}{\partial t} + \frac{v}{3} \operatorname{div}_r \boldsymbol{f}_1 + \frac{e}{3mv^2} \frac{\partial}{\partial v} (v^2 \boldsymbol{E} \boldsymbol{f}_1) = -S_{ee}(f_0) + \frac{1}{2v^2} \frac{\partial}{\partial v} \left\{ \delta_{yn} v^2 \nu \times \right.$$

$$\left. \times \left(\frac{kT}{m} \frac{\partial f_0}{\partial v} + v f_0 \right) \right\} - \sum_{i<k} \int \int q_{ik}(v, \Theta) v \left| f_0(v) F_i(v_1) - f_0(v') F_k(v_1') \right| d\boldsymbol{v}_1 d\Omega; \qquad (3)$$

$$\frac{\partial \boldsymbol{f}_1}{\partial t} + v \operatorname{grad}_r f_0 + \frac{e}{mc} [\boldsymbol{H} \boldsymbol{f}_1] + \frac{e\boldsymbol{E}}{m} \frac{\partial f_0}{\partial v} = -S_{ee}(\boldsymbol{f}_1) - \nu_{yn} \boldsymbol{f}_1 - \nu_n \boldsymbol{f}_1. \qquad (4)$$

The first term, in the right-hand member of equqtions (3) - (4), describes collisions between electrons; it is presented in detail in, for example, [17, 18].

The second term in the right-hand member of equations (3) - (4) describes elastic collisions between electrons and heavy particles. In this case the collision integral is presented, as is known, in differential form (due to the fact that in elastic collision an electron transfers to the heavy particle only a small part of its energy). Here $\delta_{yn} = 2m/M$ is the portion of energy lost on the average by an electron in one elastic collision and $\nu_{yn}(v)$ is the frequency of elastic collisions between an electron and molecules or ions (for details see, for

instance, [3] (§59).

Finally, the third term describes non-elastic collisions between electrons and heavy particles*. Here v', V_1' and v, V_1 are the velocities of electrons and heavy particles respectively before and after collision, F_i (or F_k) is the velocity distribution function of heavy particles in the state i (or k) and q_{ik} is the differential effective scattering cross section of an electron in the case of non-elastic collisions with heavy particles, accompanied by a transition of the latter from the state i to the state k. The integration is with respect to the velocities of the heavy particles dv_1 and the scattering angles $d\Omega$; summation is over all the levels i,k, **.

As was stated above, in deriving equations (3), (4), we have neglected the function χ compared with f_0 (and also, in the interelectron collision integral the function f_1^2 in comparison with f_0^2). In order to establish when this is possible, it is

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* The term describing non-elastic collisions in equation (1) is generally written in a more complex form. However, it can be shown that, with the same degree of accuracy as that with which equations (3) and (4) themselves hold true, it can be presented in the form $-\nu_H f_1$, where

$$\nu_H(v) = \sum_{i<k} v\,(N_m' + N_m'') \int q_{ik}(1-\cos\theta)\,d\Omega.$$

** It should be noted that for reasons of simplicity we shall not examine here non-elastic collisions accompanied by a change in the number of electrons in the plasma (re-combination, adhesion, dissociation, etc.). The point is that in the sufficiently rarified plasma attainable under ordinary conditions in a discharge tube as well as in the ionosphere, the effective cross section of the processes is small, and they cannot have any material influence on the electron velocity distribution; they determine only the electron equilibrium density (see [1], ch. 8, [3] §60).

necessary to make a complete expansion of the distribution function with respect to spherical functions. Let us examine, for the sake of simplicity, the case $\underline{H} = 0$ and disregard inter-electron collisions; let us also assume that the space gradient of the distribution function is directed along the z-axis parallel to $\underline{E}$. Then, expanding the distribution function of equation (1) in Legendre polynomials:

$$f(r, v, t) = \sum_{k=0}^{\infty} P_k(\cos\theta) f_k(z, v, t)$$

(θ) is the angle between $\underline{v}$ and $\underline{E}$), we arrive at the following system of equations for the functions f_0, f_1, f_2...:

$$\frac{\partial f_0}{\partial t} + \frac{v}{3}\frac{\partial f_1}{\partial z} + \frac{eE}{3mv^2}\frac{\partial}{\partial v}(v^2 f_1) = -S_{in}(f_0) - S_{el}(f_0);$$

$$\frac{\partial f_1}{\partial t} + v\left(\frac{\partial f_0}{\partial z} + \frac{2}{5}\frac{\partial f_2}{\partial z}\right) + \frac{eE}{m}\left[\frac{\partial f_0}{\partial v} + \frac{2}{5v^3}\frac{\partial}{\partial v}(v^3 f_2)\right] = -\nu_1 f_1; \quad (5)$$

$$\frac{\partial f_2}{\partial t} + v\left(\frac{2}{3}\frac{\partial f_1}{\partial z} + \frac{3}{7}\frac{\partial f_3}{\partial z}\right) + \frac{eE}{m}\left[\frac{2}{3}v\frac{\partial}{\partial v}\left(\frac{1}{v}f_1\right) + \right.$$

$$\left. + \frac{3}{7v^4}\frac{\partial}{\partial v}(v^4 f_3)\right] = -\nu_2 f_2,$$

where the collision frequencies ν_1 and ν_2 are of the same order. Hence it is seen that we can restrict ourselves to an examination of just the first two functions f_0 and f_1 (as we did in (3) and (4) above,) if the function f_2 is far smaller than f_0 and if at the same time the derivative $\partial f_2/\partial z$ is far smaller than $\partial f_0/\partial z$. Expressing the function f_2 in terms of f_0, these necessary conditions may be presented in the form

$$\left(\frac{eEl}{mv^2}\right)^2 \ll 1; \quad (6a)$$

$$\frac{\partial f_0}{\partial t} \ll \nu f_0; \quad (6b)$$

$$l\frac{\partial^2 f_1}{\partial z^2} \ll \frac{\partial f_0}{\partial z} \qquad (6c)$$

(here $l = v/\nu$ is the length of the free path of the electron). If we proceed to average values, then the first of these conditions is identical with the requirement

$$\delta \ll 1$$

where δ is the portion of energy lost on the average by an electron as a result of one collision with a heavy particle. This condition is always well fulfilled in a molecular plasma (see, for instance, Table 1 in section 2 of the present article.) The second and third conditions (6b) and (6c) require respectively that the electron energy should not change significantly during the free run of the electrons and that the electron current should not change significantly over the length of that run. Taking into account the magnetic field and inter-electron collisions does not change these conditions of applicability of the system of equations (3), (4)*.

Let us pass to the analysis of equations (3), (4). First of all, observe that non-elastic collisions have been accounted for in the equation for the directional part of distribution function f_1 simply by replacing the frequency of elastic collisions ν_{yn} with the total collision frequency $\nu = \nu_{yn} + \nu_H$. Therefore the expressions for function f_1, obtained for the case of elastic

* The same conditions (6) are necessary when only elastic collisions between electrons and heavy particles occur, in the plasma; then δ should merely be replaced by $\delta_{yn} = 2m/M$. The case of elastic collisions has been examined in detail by Davydov [6]. However, it should be noted that condition (6b) was omitted in that work and (6c) was written in the form $l\,\partial f_1/\partial z \ll f_0$, which is incorrect.

collisions alone (e.g. see [8], fully preserve their form even after taking non-elastic collisions into account. Thanks to that, the task of finding the electron distribution function, in fact, reduces to integrating one equation for the function f_0.

The first term of the right-hand member of this equation, governed by inter-electron collisions, is of the order of $\nu_e f_0$, where ν_e is the frequency of inter-electron collisions. The remaining terms, describing collisions between electrons and heavy particles, are of the order $\delta\nu f_0$. It is clear that, depending on the value of the ratio $\nu_e/\delta\nu$, the form of function f_0 will be determined either by the inter-electron collisions (if $\nu_e/\delta\nu \gg 1$), or by collisions with heavy particles if $\nu_e/\delta\nu \ll 1$) [18]. In the case of collisions with ions, examined, for instance, in [7], this ratio is always much greater than unity (since ν_e and ν_i are of the same order and $\delta \ll 1$). However, in the case of collisions with molecules

$$\frac{\nu_e}{\delta\nu_m} = \frac{3\pi}{2\sqrt{2}}\frac{e^4}{\delta(kT_e)^2\delta_m}\frac{N}{N_m}\ln\left(0{,}4\frac{kT_e}{e^2N^{1/3}}\right) \approx$$

$$\approx 10^6\frac{N}{N_m}\left(\frac{300^\circ}{T_e}\right)^2\left(\frac{2\cdot10^{-5}}{\delta}\right)\left(\frac{4\cdot10^{-16}\text{см}^2}{q_m}\right). \qquad (7)$$

This ratio can be both greater and smaller than unity, depending on the degree of ionization of the plasma*.

In this connection two essentially different cases may be distinguished in solving equation (3): the case in which the degree of ionization of the plasma is significant, so that $\nu_e/\delta\nu_m \gg 1$, and the case of a weakly ionized plasma, when

* In the ionosphere $\nu_e/\delta\nu_m \ll 1$ in layer D and in the lower part of layer E (up to altitudes of the order of 90 ÷ 100 km). In higher levels, on the other hand, $\nu_e/\delta\nu_m \gg 1$.

$\nu_e / \delta \nu_m \ll 1$. Both these cases will be examined below.

First, however, let us note another peculiarity of the molecular plasma, thanks to which equation (3) can be significantly simplified. When electrons with energy of the order of less than 1 ev are in non-elastic collision with the molecules of a plasma, generally speaking both rotational and vibrational levels of the molecules may be excited. However, an analysis of experimental data shows that when $\mathcal{E} \lesssim 1$ ev the main part of the losses is occasioned by collisions leading only to the excitation of rotational levels* . Let us now take into account the fact that

* As a matter of fact, in the experimental measurement of δ the mean electron energy varies from 0.05 ev to 1 - 2 ev and more. The energy of excitation of rotational levels is then always small in comparison with the energy of the electrons; therefore the losses for the excitation of rotational levels can depend only very slightly on $\mathcal{E}$. The energy of excitation of vibrational levels, on the other hand, is of the order of 0.2 - 0.5 ev. It is understandable that at minimal values $\mathcal{E} \sim 0.005$ ev the losses for the excitation of vibrational levels are insubstantial. They begin to exert a serious influence only at high electron energies: in this region δ should increase sharply. Experimentally δ varies only slightly up to energies of the order of 1 - 2 ev and only subsequently does it increase sharply (see [4,5]). On the basis of the above it has been concluded that at electron energies of less than 1 ev, the main part is played by losses for the excitation of rotational levels [9]. This assumption has subsequently been confirmed by a special experiment, conducted by Hall [10]. Of course it needs, further proof. The most convincing proof would be a direct measurement of cross sections for excitation of the vibrational and rotational levels of molecules by slow electrons. There is also the possibility of an indirect check by comparing the results of an experimental investigation of electron velocity distribution in a molecular plasma with those of theory (see section 3).

the energy of the rotational quanta is small in comparison with the mean energy of the electrons. This means that in a molecular plasma electrons with low energies (0.01 ev $\leqslant \mathcal{E} \leqslant$ 1 ev) lose only a small portion of their energy even in non-elastic collisions. Utilizing this basic fact, it is easy to convert to differential form the integral term which in equation (3) describes non-elastic collisions between electrons and heavy particles.

In point of fact, it is obvious that in the case in which electrons lose in one collision only a small part of their energy, the collision integral for the symmetrical part of the distribution function f_0 can be put in the form:

$$S = -\operatorname{div} j_v = -\frac{1}{v^2}\frac{\partial}{\partial v}\{v^2 j_v\},$$

where j_v is the flux of particles through unit area of a spherical surface v in the velocity (space) caused by the corresponding collisions (cf. [17]). In the case under discussion S is the integral term in equation (3), describing non-elastic collisions; the following expression is valid for the flux j_{vH}:

$$j_{vH} = \sum_{i<k} \int\int q_{ik}(v, \Theta)\, v\,(v' - v)\left\{ f_0(v) F_i(v_1) - f_0(v') F_k(v_1')\right\} dv_1 d\Omega .$$

Here, as in the usual Boltzmann collision integral, $q_{ik}(v,\Theta)$ is the cross section for a non-elastic collision, leading to the emission of a quantum $\hbar\omega_{ik}$, $F_i(v_1)$ is the molecular velocity distribution function. The integration is with respect to the molecular velocities dv_1 and the scattering angles $d\Omega$; the summation is over all the levels i,k. Taking into account the fact that an electron loses only a small part of its energy in one collision (i.e. $|v - v'| \ll v$), the expression in curly brackets may be modified somewhat:

$$\{f_0(v)F_i(v_1)-f_0(v')F_k(v_1')\}=(v'-v)\frac{\partial f_0}{\partial v}\frac{F_i(v_1)+F_k(v_1')}{2}+$$
$$+f_0(v)\left[F_i(v_1)-F_k(v_1')\right].$$

Let us also take into account that in a non-elastic collision an electron loses energy only in exciting a molecule; in consequence $(v' - v) = h \cdot \omega_{ik}/mv$. The expression for the flux j_v, and consequently, for the non-elastic collision integral also is now converted to the differential form sought:

$$J_{vn}=\sum_{i<k}^{'}\nu_{ik}\left\{\left(\frac{h\omega_{ik}}{mv}\right)^2\frac{1+N_m^k/N_m^i}{2}\frac{\partial f_0}{\partial v}+\frac{h\omega_{ik}}{mv}\left(1-\frac{N_m^k}{N_m^i}\right)f_0\right\}. \qquad (8a)$$

Here N_m^i (or N_m^k) is the number of molecules in the state i (or k), ν_{ik} is the total frequency of excitations of the level $h\omega_{ik}$ as a result of collisions with an electron having a velocity v:

$$\nu_{ik}(v)=2\pi N_m^i v\int_0^{\pi}q_{ik}(v,\Theta)\sin\Theta\,d\Theta.$$

Let us now assume that the molecules are distributed with respect to level in accordance with the Maxwell - Boltzmann law: $N_m^k/N_m^i = \exp\{-h\omega_{ik}/KT\}$ (i.e. collisions with electrons do not substantially alter the number of excited molecules) and let us also take into account the fact that at room temperatures not only the electron energy but also the mean energy of the molecules in a molecular plasma is much greater than the rotational quanta ($kT \gg h \cdot \omega_{ik}$). Then the expression for the non-elastic collision integral in a molecular plasma will finally assume the following form:

$$S_n=-\frac{1}{2v^2}\frac{\partial}{\partial v}\left\{v^2Q_n(v)\left[\frac{kT}{m}\frac{\partial f_0}{\partial v}+vf_0\right]\right\}, \qquad (8b)$$

where $Q_H(v)$ is the portion of energy lost by an electron in unit time in non-elastic collisions:

$$Q_H(v) = \sum_{i<k} \nu_{ik} \frac{2h\omega_{ik}}{mv^2} \left(1 - \frac{N_m^k}{N_m^i}\right) = \sum_{i<k} \nu_{ik} \frac{2(h\omega_{ik})^2}{mv^2 kT}. \quad (9)$$

Thus, in a molecular plasma, for conditions under which the electron energy is small ($\mathcal{E} \lesssim 1$ ev), it is possible to use the simple expression (8b) for the non-elastic collision integral. In what follows these conditions are assumed to be fulfilled*.

2. The case of a significant degree of ionization (Maxwellian distribution).

In the case in which the degree of ionization of the plasma is considerable, i.e. $\nu_e \gg \delta\nu_m$, the form of the function f_0 ought to be determined by inter-electron collisions. Therefore, as a first approximation (expanding in powers of the parameter $\delta\nu_m/\nu_e$) the function f_0 is Maxwellian. Consequently, the problem reduces to finding merely the electron temperature**. The latter, as is readily seen, is determined from the equation:

$$\frac{dT_e}{dt} + \delta_{ee}\nu_{ee}(T_e - T) = \frac{2}{3kN} Ej. \quad (10)$$

* It should be emphasized that the above simplification of the collision integral (to formula (8a)) utilizes only the smallness of the portion of energy lost by an electron in one non-elastic collision. In this context it should be mentioned that the direct experiments of Ramien, Harries and Herts [11] show that losses of electron energy in hydrogen and nitrogen at higher energies ($\mathcal{E} \sim 5 - 7$ ev) are governed by the excitation of only the lower oscillating levels; consequently, the portion of energy lost by an electron in such collisions is also small (of the order of 0.03 - 0.1). Therefore it is possible that the method adopted here is valid not only up to $\mathcal{E} \sim 1$ ev, but also for higher energies.

** Only the case of a homogenous plasma is examined below.

Here $\nu_{э\phi\phi}$ and j are the effective frequency of collisions and the electron current, defined by the relationships:

$$\nu_{э\phi\phi} = \frac{\sqrt{2}}{3\sqrt{\pi}}\left(\frac{m}{kT_e}\right)^{3/2}\int_0^\infty \nu(v)v^4\exp\left\{-mv^2/2kT_e\right\}dv;$$

$$j = e\int v f(dv) = \frac{4\pi}{3}e\int_0^\infty v^3 f_1 dv. \qquad \text{(11a)}$$

The expressions for $\nu''_{э\phi\phi}$ and j are the same as in the case of elastic collisions only. They have been analyzed already (see, for instance [3, 7, 8]).

Further $\delta_{э\phi\phi}$ in equation (10) is the portion of energy lost on the average by an electron in one collision*.

$$\delta_{э\phi\phi} = \delta_{уп} + \frac{\sqrt{2}}{3\sqrt{\pi}\nu_{э\phi\phi}}\left(\frac{m}{kT_e}\right)^{3/2}\int_0^\infty Q_u(v)v^4\exp\left\{-mv^2/2kT_e\right\}dv, \qquad \text{(11b)}$$

where the value quantity $Q_u(v)$ is determined in accordance with (9). This, of course, should be known in order to be able to calculate $\delta_{э\phi\phi}$. However, the fraction of energy $Q_u(v)$ cannot at present be found from (9), because the cross sections for non-elastic collisions between slow electrons and molecules are not known (see [2]).

The quantity $\delta_{э\phi\phi}$ and its depoadence on temperature have often been measured experimentally. The results for hydrogen,

* Note that, as is made clear by equation (10), $\delta_{э\phi\phi}$ is strictly, the mean fraction of surplus energy of the electron $3/2\ kT_e - 3/2\ kT$. The value $\delta_{э\phi\phi}$ is connected with mean fraction of the total electron energy η by the relationship $\delta_{э\phi\phi} = \eta T_e/(T_e - T)$; at high electron energies, when $T_e \gg T$, the values of $\delta_{э\phi\phi}$ and η coincide. Values of η are commonly used in experimental work.

oxygen and nitrogen are given in Table 1*.

Table 1.

Values of $\delta_{эфф} \cdot 10^3$

$T_e \cdot 10^{-3}$ (°K)	H_2	O_2	N_2	Воздух	Atmosphere h=100 км	h=200 км	h=300 км
0,5	2,3	—	—	—	—	—	—
1	2,5	3,7	0,47	0,89	0,86	0,08	0,06
2	2,2	6,7	0,36	1,2	1,2	0,12	0,06
3	2,2	3,6	0,33	1,6	1,5	0,16	0,06
4	2,5	9,05	0,32	1,7	1,6	0,18	0,06
5	3,0	8,7	0,34	1,7	1,6	0,22	0,06
6	3,4	8,2	0,38	1,7	1,6	0,26	0,07
7	3,9	7,7	0,45	1,7	1,6	0,32	0,07
8	4,4	7,2	0,60	1,7	1,6	0,43	0,08
9	4,85	6,8	0,82	1,8	1,7	0,60	0,09
10	5,3	6,6	1,15	2,0	2,0	0,85	0,11
12	6,1	7,7	2,40	3,2	3,1	1,8	0,23
15	7,2	21	9,8	11	10,6	7,7	1,13

* The values of $\delta_{эфф}$ given here are from the data of Crompton and Sutton [5], subsequently confirmed by Hall [5, 10]. The character of the dependence of $\delta_{эфф}$ on T_e, according to these data, essentially coincides with the dependence obtained by earlier authors [4], although the discrepancy in absolute values is quite significant. The temperature of plasma in the experiment was $T \approx 290°$; a special check at lower T did not reveal any change in $\delta_{эфф}$ [10].

It should be emphasized that the introduction of the quantity $\delta_{эфф}$ is meaningful only in the case in which the electron velocity distribution is Maxwellian, i.e. when the condition $\nu_e/\delta\nu_0 \gg 1$ is fulfilled in the experimental work. Therefore the authors calculate $\delta_{эфф}$ for two cases; for Maxwellian and Druyvestein distribution. The discrepancy between values of $\delta_{эфф}$ proved to be small (up to 10 - 20%).

When the gas is a mixture of two(or more) gases, the values of $\delta_{э\phi\phi}$, as is readily seen, can be calculated in the following manner:

$$\delta_{э\phi\phi} = \delta_{э\phi\phi 1}\frac{\nu_{э\phi\phi 1}}{\nu_{э\phi\phi 1}+\nu_{э\phi\phi 2}} + \delta_{э\phi\phi 2}\frac{\nu_{э\phi\phi 2}}{\nu_{э\phi\phi 1}+\nu_{э\phi\phi 2}}. \qquad (12)$$

Here $\delta_{э\phi\phi 1}$, and $\nu_{э\phi\phi 1}$, are the portions of energy lost by an electron and the collision frequency for gas 1; correspondingly, $\delta_{э\phi\phi 2}$, $\nu_{э\phi\phi 2}$ for gas 2. The results of calculating $\delta_{э\phi\phi}$ for air from formula (12) are given in Table 1 and in Fig. 1. The same figure contains the results of experimental determinations of $\delta_{э\phi\phi}$ for air (Huxley, Crompton and Sutton [5], Huxley and Zaazou [5]). It is evident from the figure that the

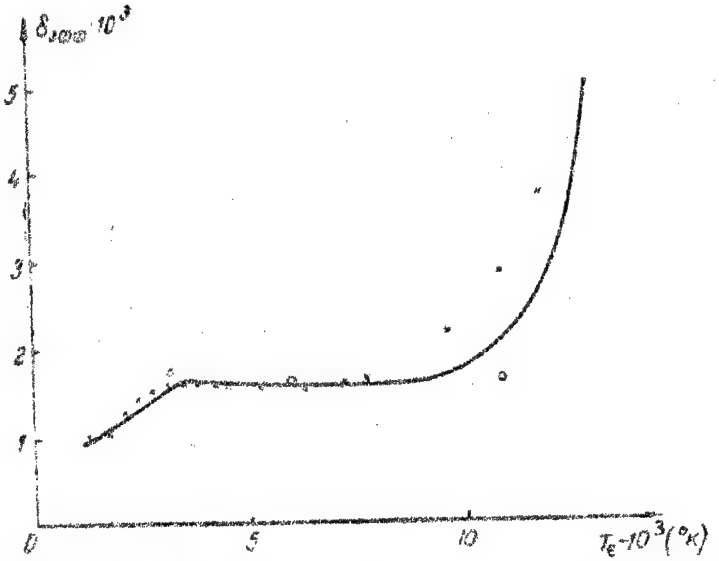

Fig. 1. Dependence of $\delta_{э\phi\phi}$ on T_e for air; x, ⊙ are the experimental results [5].

the measured values are in good agreement. Table 1 also contains values of $\delta_{э\phi\phi}$ for the ionosphere at altitudes of 100, 200 and 300 km, calculated from formula (12) (composition of the ionosphere at these altitudes is assumed to be in agreement with the

model proposed in [12] on the basis of data obtained in rocket experiments*).

3. The case of weakly ionized plasma.

In the case of a weakly ionised plasma ($\nu_e \ll \delta\nu_m$) interelectron collisions are unimportant and can be neglected in equation (3). If, moreover, we bear in mind the trnasformation of (8b) for the non-elastic collision integral, it is easy to see that equation (3) in fact coincides with the usual equation for the function f_0 in the case of elastic collisions alone: one need merely exchange $\delta_{yn} = 2m/M$ for $\delta(v)$, where

$$\delta(v) = \left| \beta_{yn} \nu_{yn}(v) + Q_n(v) \right| \nu^{-1}(v). \qquad (13)$$

The solution of equation (3) also coincides with the solutions of the equation for the case of elastic collisions alone, obtained in [6,8,14,15]: again it is sufficient to exchange δ_{yn} for $\delta(v)$. Proceeding in this manner, we find that in the case of a strong steady electric field

$$f_0 = C \exp\left\{ -\frac{3m^2}{2e^2E^2} \int_0^v v\nu^2(v)\delta(v)dv \right\}, \qquad (14)$$

where the constant C is determined by the normalization condition.

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* With the aid of a simple approximation of values of the presented $\delta_{эфф}$ in Table 1 for the region of low electron temperatures, for the region of the ionosphere responsible for the cross-modulation of radio waves (h ~ 90 km, $T_e \sim 250°$) we get a value for $\delta_{эфф} \approx 0.9 \times 10^{-3}$. At the same time experimental data on cross-modulation [13] appear to indicate a somewhat higher value of $\delta_{эфф}$, of the order of 2×10^{-3}. In view of this the experimental determination of $\delta_{эфф}$ for air at low electron temperatures ($T_e < 1000°$) would be a particular great interest.

In the absence of non-elastic collisions, when $\delta(v) \equiv \delta_{yn}$, this distribution function naturally coincides with the Druyvestein distribution [14]. In the case of a variable electric field $E = E_0 \cos \omega t$, the frequency ω of which is much greater than $\delta\nu_{эфф}$

$$f_0 = C \exp\left\{ - \int_0^v \frac{mv\,dv}{kT + \left[e^2E_0^2/3m^2(v)(\omega^2 + \nu^2)\right]} \right\}. \qquad (15)$$

In the absence of non-elastic collisions (15) coincides with the distribution function obtained by Margenau [15].

The form of the distribution function f_0 in a molecular plasma, as is clear from (14) - (15), depends essentially on the form of the function $\delta(v)$. The latter, as has been shown above, cannot at present be calculated from the (9), because the cross sections for collisions between slow electrons and molecules are unknown. However, the function $\delta(v)$ can be found in a different way, viz. by using the results of experimental measurements of the dependence of $\delta_{эфф}$ on T_e. In point of fact, in accordance with formula (11b):

$$\int_0^\infty \delta(v)\,\nu\,v^4 \exp\left\{ - mv^2/2kT_e \right\} dv = \int_0^\infty \delta(v) K(v)\,dv = \qquad (16)$$

$$= \frac{3\sqrt{\pi}}{\sqrt{2}} \left(\frac{kT_e}{m}\right)^{5/2} \delta_{эфф}(T_e)\,\nu_{эфф}(T_e).$$

The function in the right-hand member of this expression has been determined experimentally; the Kernel $K(v) = \nu v^4 \exp\left\{-mv^2/2kT_e\right\}$ is also known. Consequently expression (16) can be considered as an integral equation with respect to the function $\delta(v)$; as is known, its solution can be found by reducing equation (16) to a system of linear algebraic equations.

Note that in our case it is nearly always possible to simplify equation (16), by converting it into differential form (by the Fokker - Planck method); the solution sought (assuming that ν is proportional to v) is then written in the form

$$\delta(v) = \delta_{\text{эфф}}(T_e^*) - 0{,}086 T_e^* \delta'_{\text{эфф}}(T_e^*) - 0{,}173\, T_e^{*2} \delta'_{\text{эфф}}(T_e^\circ), \quad (17)$$

where $T_e^\circ = (T_e^\circ)(v) = 128\ mv^2/225\varkappa k \approx 118^\circ\ (v/10^7)^2$. . The condition of applicability of solution (17) is the smallness of the correction terms compared with the main (first) term.

The results of calculating the function δ (v) for hydrogen, oxygen, nitrogen and air are given in Table 2, and, graphically, in Fig. 2.

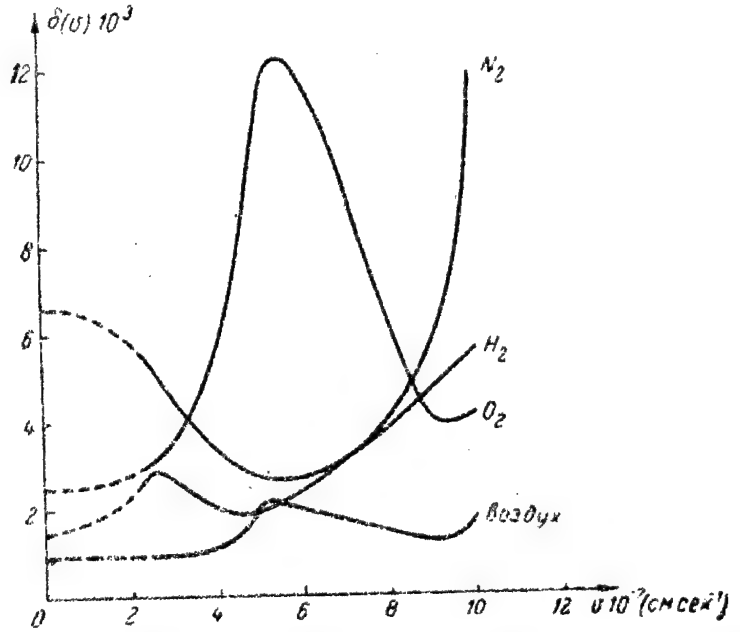

Fig. 2 Dependence of δ on v for N_2, O_2, H_2 and air. In the case of nitrogen values of δ (v) x 10^{-4} are plotted along the ordinate axis.

Substituting the function δ (v) thus found in (14),

(15), or in other expressions for f_0 [6, 8], it is possible to calculate the electron distribution function in a molecular plasma. Fig. 3 shows the results of a corresponding calculation

Table 2.

$\varepsilon \cdot 10^{-3}$ (eV)	$k(\varepsilon) \cdot 10^{17}$			
	H_2	O_2	N_2	air
1,0	—	—	—	—
1,5	1,9	—	—	—
2,0	2,3	2,7	0,53	0,69
2,5	2,8	3,6	0,52	0,88
3,0	2,6	3,5	0,44	0,87
3,5	2,2	4,3	0,38	0,94
4,0	2,0	5,6	0,33	1,1
4,5	1,8	7,8	0,30	1,4
5,0	1,9	10,8	0,27	1,9
5,5	2,1	12,2	0,25(5)	2,1
6,0	2,4	11,7	0,27	1,9(5)
6,5	2,7	10,7	0,28	1,8
7,0	3,1	9,2	0,31	1,7
7,5	3,4	7,8	0,34	1,5(5)
8,0	3,8	6,2	0,40	1,4
8,5	4,2	4,8	0,47	1,3
9,0	4,7	4,0	0,57	1,2
9,5	5,1	3,9	0,74	1,3
10,0	5,6	4,1	1,2	1,8

of the electron distribution function in hydrogen in a high-frequency electric field (see (15) for $\omega^2 \gg \nu^2$). In f_0 is plotted along the ordinate and $v^2/\bar{v}^2$ along the abscissa, where $\bar{v}^2 = 2\varepsilon/m$ is the mean square electron velocity. The broken line represents the Maxwellian distribution function. It is evident from the figure that in the case under consideration, the function f_0 does not deviate much from the Maxwellian function; the deviations increase with increase in the mean electron energy. The results of an analogous calculation for oxygen and nitrogen are given in Fig. 4.

Using the expressions obtained for the distribution function, it is possible to calculate the mean electron energy ε, the conductivity σ and the dielectric constant of the

plasma ε*:

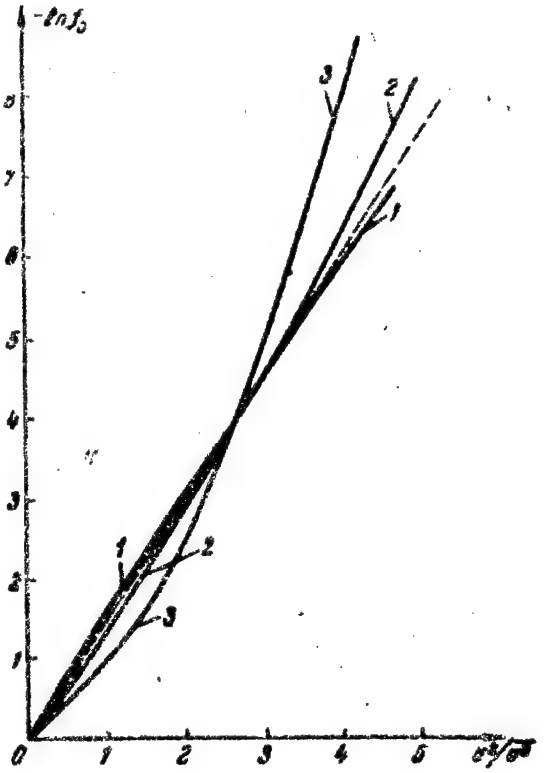

Fig. 3 Dependence of $-\ln f_0$ on $v^2/\bar{v}^2$ for hydrogen in a high-frequency field for different values of the mean electron energy: 1 - $\varepsilon = m\bar{v}^2/2$ = = 0.1 ev, curve 2 - ε = 0.3 ev, curve 3 - ε = 1 ev.

* For reasons of simplicity, the expressions for σ and ε are for the absence of a magnetic field. In the presence in the plasma of a steady magnetic field they may be transformed in the usual way (see, for instance, [3]): components of the tensors σ and ε in the direction z, parallel to H, preserve the form of (18): in order to calculate σ_{xx} or σ_{yy} it is necessary to replace $1/(\omega^2 + \nu^2)$ in (18) with

$$\frac{1}{2}\left\{\frac{1}{(\omega-\omega_H)^2+\nu^2}+\frac{1}{(\omega+\omega_H)^2+\nu^2}\right\}$$

$$\mathcal{E} = \frac{2\pi m}{N} \int_0^\infty v^4 f_0 dv ;$$

$$\sigma = \frac{4\pi e^2}{3} \int_0^\infty \frac{\nu v^4 f_0 dv}{kT(\omega^2+\nu^2) + e^2 E_0^2/3m\,\delta(v)} ; \qquad (18)$$

$$\varepsilon = 1 - \frac{16\pi^2 e^2}{3} \int_0^\infty \frac{v^4 f_0 dv}{kT(\omega^2+\nu^2) + e^2 E_0^2/3m\,\delta(v)} .$$

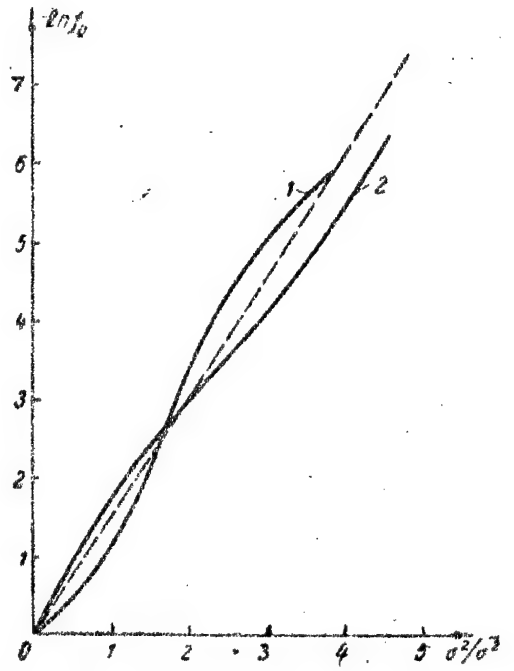

Fig. 4 Dependence of $-\ln f_0$ on $v^2/\bar{v}^2$ for oxygen (1) and nitrogen (2) in a high-frequency electric field for $\mathcal{E}$ -0.3 ev.

As an illustration, Fig. 5 shows the results of the corresponding calculation of $\mathcal{E}$ and σ for a hydrogen plasma in a high-frequency electric field (solid curves).

4. On the possibility of using the elementary equations for calculating the electron current and the mean energy of the electrons.

In the simplest case, when the frequency of collision of the electron and the fraction of energy lost by it in one collision do not depend on its velocity ($\nu \neq \nu(v)$, $\delta \neq \delta(v)$), the solution of the system of equations (3), (4) has the following form:

$$f_0 = C \exp\{-mv^2/2kT_e\}; \qquad f_1 = -u\,\partial f_0/\partial v.$$

Here T_e is the temperature of the electrons and u is their mean directional velocity, defined by the equations:

$$\frac{dT_e}{dt} + 2\nu(T_e - T) = \frac{2}{3k} eEu;$$
$$\frac{du}{dt} + \nu u = \frac{e}{m}\left\{E + \frac{1}{c}|uH|\right\}. \tag{19}$$

Mean electron energy and electron current are related to T_e and u by the expressions:

$$\mathcal{E} = \frac{3}{2} kT_e, \qquad j = eNu. \tag{20}$$

As we have seen above, in a molecular plasma ν and δ depend on v. Therefore, the simple expressions (19) and (20) for the mean energy of the electrons and the electron current are, generally speaking, inapplicable in this case. However, in the case of only a weak dependence of ν and δ on v, the latter may prove to be not very important. In such cases, by replacing ν and δ in equations (19) with their mean or effective values $\nu_{эфф}(T_e)$ and $\delta_{эфф}(T_e)$, which it is natural to determine in accordance with formulas (11), we shall obtain a system of simple equations for T_e and u; these can be used for calculating approximate values of $\mathcal{E}$ and j. It should be noted that these

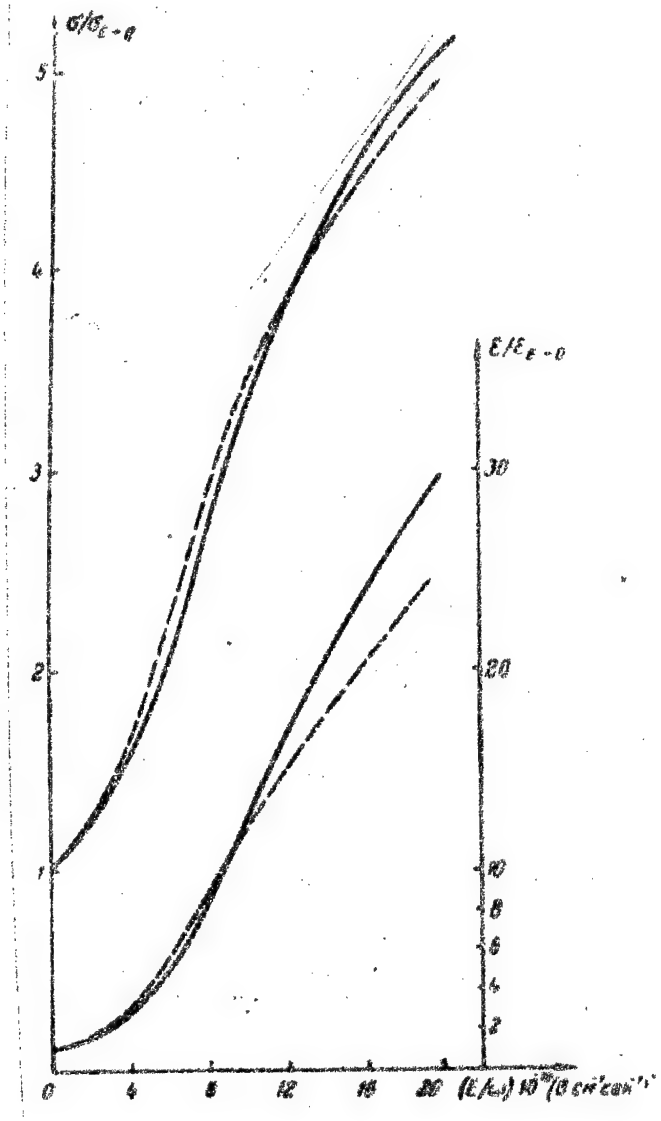

Fig. 5 Dependence of $\varepsilon/\varepsilon_{E\to 0}$ and $\sigma/\sigma_{E\to 0}$ on E/ω in hydrogen in a high-frequency field. Broken lines represent the results of calculations in accordance with the formulas of the elementary theory.

equations can be arrived at directly by starting from the elementary kinetic theory, without bringing in the question of the dis-

tribution of electron velocities (see, for instance, [3] §55): this is why they are commonly called "elementary".

As was mentioned above, solving the elementary equations only gives approximate values of $\mathcal{E}$ and j. In order to evaluate the admissible error, it is necessary to compare them with the results of an accurate kinetic calculation; where the error is large, the elementary theory is inapplicable. A corresponding examination shows that the results of calculating $\mathcal{E}$ and j by the elementary and the accurate kinetic theories do not diverge significantly, if the following conditions are fulfilled:

$$D_\nu \frac{\nu_{эфф}^2}{\omega^2+\nu_{эфф}^2} \ll 1; \tag{21a}$$

$$\frac{1}{2}\frac{T_e}{\nu_{эфф}}\frac{d\nu_{эфф}}{dT_e}\frac{\gamma(T_e-T)}{\gamma(T_e-T)+T_e} \ll 1; \tag{21b}$$

$$\frac{1}{2}\frac{T_e}{\delta_{эфф}}\frac{d\delta_{эфф}}{dT_e}\frac{\gamma(T_e-T)}{\gamma(T_e-T)+T_e} \ll 1. \tag{21c}$$

where

$$\gamma=\frac{T_e}{\delta_{эфф}}\frac{d\delta_{эфф}}{dT_e}+\frac{2\nu_{эфф}^2}{\omega^2+\nu_{эфф}^2}\frac{d\nu_{эфф}}{dT_e},$$

and D_ν is the relative dispersion of the collision frequency:

$$D_\nu=\frac{\sqrt{2}}{3\sqrt{\pi}}\frac{1}{\nu_{эфф}^3}\left(\frac{m}{kT_e}\right)^{3/2}\int_0^\infty(\nu-\nu_{эфф})^2 v^4\exp\left|-mv^2/2kT_e\right|dv. \tag{22}$$

In the case of Maxwellian distribution (a significant degree of ionization of the plasma), only the first condition in (21a) need be fulfilled. In particular, it is always satisfied at high frequencies of the electric field ($\omega^2 \gg \nu^2_{эфф}$). In the case of a low frequency ω, it is necessary that $\nu_{эфф}$ should not depend strongly on the electron temperature: if $\nu_{эфф} \sim T_e^\alpha$, then only when $-1 \leqslant \alpha \leqslant 1$ is the error in the elementary calculation small ($\Delta\mathcal{E} \lesssim 50\%$; $\Delta\sigma \lesssim 50\%$). It should be noted that the error $\Delta\varepsilon$, generally speaking, is considerably larger; it

is small only when $-0.5 < \alpha < 0.5$.

In a weakly ionized plasma conditions (21b) and (21c) should also be fulfilled*. It is also necessary that $\nu_{\varepsilon\phi\phi}$ and $\delta_{\varepsilon\phi\phi}$ be weakly dependent on T_e; the error in calculating σ and $\mathcal{E}$ is small if $-0.25 \leqslant \alpha \leqslant 1$. It is important that the error, admissible in the elementary calculation, increases rapidly in the region where $\delta_{\varepsilon\phi\phi}$ decreases with increase in T_e; this is also evident from conditions (21b) and (21c)**.

For the molecular gases examined above (H_2, O_2, N_2, air) when the electron energy is small $\nu_{\varepsilon\phi\phi}$ is proportional to $T_e^{0.3 \div 0.9}$ (see [4, 5]) and $\delta_{\varepsilon\phi\phi}$ depends even more weakly on T_e (see Table 1.). Consequently in these cases conditions (21) are fulfilled, so that the elementary theory can be used to calculate $\mathcal{E}$ and j. However, it should be mentioned that at low frequencies ($\omega^2 \lesssim \nu^2_{\varepsilon\phi\phi}$) kinetic corrections can be significant, particularly so in calculating ε ; at high frequencies

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* It must be mentioned, that conditions (21b), (21c) are strictly valid only in the region, where the dependence of $\nu_{\varepsilon\phi\phi}$ and $\delta_{\varepsilon\phi\phi}$ on T_e is monotone. In the vicinity of points where the monotony is disturbed, it is necessary that additional conditions $D_\nu \ll 1$, $D_\delta \ll 1$, where D_ν and D_δ are the relative dispersions of ν and δ (see (22)), should be fulfilled.

** In cases in which $\nu_{\varepsilon\phi\phi}$ or $\delta_{\varepsilon\phi\phi}$ fall sharply with increase in T_e (if $\alpha < -1/2$), the elementary and the exact kinetic theories even lead to qualitatively different results. As is shown by the exact theory, the effect of instability and electron temperature hysteresis, obtained in the elementary theory, occurs only where the degree of ionization of the plasma is considerable; in weakly ionized plasmas there is no instability (see [16]).

they are small. For example, Fig. 5 shows broken curves, representing $\mathcal{E}$ and σ in a hydrogen plasma for $\omega^2 \gg \nu_{эфф}^2$, calculated in accordance with the elementary theory formulas (the solid curves are the results of exact kinetic calculations). It is evident from the figure that in the example reviewed the broken and solid curves diverge at the most by 15%.

In conclusion, the author expresses his indebtedness to V.L. Ginzburg for his interest in the work.

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ON BACK - SCATTERING OF SHORT RADIO-WAVES IN THE LOWER IONOSPHERE

Pages 370-373

By O.I. Yakovlev& V.I. Bocharov

The theory of the scattering of radio waves at statistical inhomogeneities in media is applied to the problem of back -scattering of radio waves in the lower ionosphere. The dissipated power and the correlation coefficient of signals, received by spaced antennas, are calculated. The results of preliminary experimental investigations of back-scattering of radio waves at a frequency of 12 Mc are presented.

It is known that the lower ionosphere is a statistically inhomogeneous medium which scatters radio waves. Apparently, the first detailed investigation of the scattering of short waves by the lower ionosphere was carried out by Eckersley [1]. Assuming that the scattering of radio waves can insure long-distance USW communications Bailey [2] has made a detailed study of the ionospheric scattering of meter waves. Booker and Gordon [3], and later Villars and Weisskopf [4], have developed the theory of the scattering of radio waves at atmospheric inhomogeneities. These variants of the theory are based on definite assumptions about statistical regularities of the scattering media. The question of the validity of the regularities assumed remains open. A comparison between theory and the experiments of Eckersley and Bailey is difficult, because these experiments permit the determination of the effective scattering diameter for only a limited region of angles of scatter. Under

these conditions different variants of the theory of scattering can be made to correspond with experiment by the appropriate choice of the parameters entering into the theory.

In this context it is of interest to observe and to investigate the back-scattering of short waves by the lower ionosphere. In vertical sounding of the ionosphere at frequencies above the critical, it is possible to receive signals governed by at least three mechanisms. In the first place, there is the possibility of the partial reflection of radio waves from irregularly occurring regions with a large electron density gradient; secondly, the appearance of signal "splashes", caused by meteoric ionization, and, finally, there should be signals due to the back-scattering of radio waves. Signals of the last type should be characterized by their regular presence, a rapidly fluctuating weak level and non-correlation of the instantaneous values of the signals, received by spaced antennas.

In order to calculate the dissipated power at the receiver input and the correlation coefficient, we shall examine the idealized scheme of back-scattering shown in Fig. 1. A plane, statistically inhomogeneous layer of thickness b at altitude h is irradiated by a spherical wave. Let a transmitter of power P generate pulses of duration τ and let the antenna system have an amplification factor K. It is easily shown that at the receiver input the power of the scattered signal is equal to

$$W = \frac{PK^2\lambda^2\sigma}{16\pi^2 J}, \qquad (1)$$

where $J = \int r^{-4} dv$, σ is the effective diameter of back-scattering of the radio waves. By assuming the directional diagrams of the antennas to be sufficiently broad, we disregard their part in expression (1). The integration in formula (1) is with respect to the volume, important in back-scattering. This volume is defined by the intersection of the plane layer and a spherical zone of thickness $a = c\tau/2$; this volume is shaded in Fig. 1). J has the following values:

$$J = \pi a^2/h^3 \qquad (a \leqslant b);$$

$$J = \pi b\,[h(2a - b) + a^2] / h\,(h + a)^2 (h + b) \qquad (a \geqslant b). \qquad (2)$$

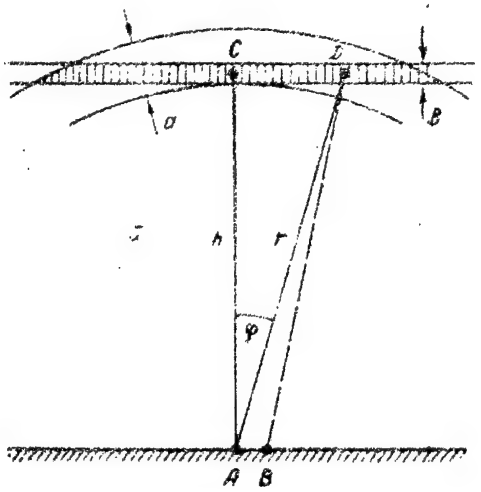

Fig. 1.

It is evident from the above expressions that for a pulse duration $\tau < 2b/c$ the power received is proportional to the square of the duration of the pulse; when the condition $\tau > 2b/c$ is fulfilled, the signal received depends little on the duration of the pulse. It should be noted that, in contrast with the case of wave scattering at small angles, the power, scattered back, depends on the duration of the pulse. Formulas (1) and (2) give a simple relationship between the power received, the effective

diameter of back-scattering and the parameters of the layer.

In determining the signal correlation coefficient we shall consider that the pulse duration is sufficiently great so that there is no limitation of the volume important for back-scattering and that the antennas have broad directional diagrams. The correlation coefficient for the ionospheric back-scattering of short radio-waves can be calculated following the method used in [5] for the analogous calculation of tropospheric scattering of ultrashort waves. In our case the correlation coefficient is determined from the expression

$$V_1 V_2^* \sim \int r^{-4} e^{iks\sin\varphi'} dv, \qquad (3)$$

in which s = AB; the sense of the angle φ' is clear from Fig. 1. It follows from formula (3) that

$$V_1 V_2^* \sim \int_0^\infty (\rho^2 + h^2)^{-2} \rho \exp\left(iks\rho / \sqrt{\rho^2 + h^2}\right) d\rho. \qquad (4)$$

Here ρ is the distance CD (see Fig. 1). We obtain the expression for the correlation coefficient directly from (4):

$$R(s) = 2\alpha^{-2}(2 + \alpha^2 - 2\alpha\sin\alpha - 2\cos\alpha)^{1/2}. \qquad (5)$$

In formula (5) $\alpha = 2\pi s/\lambda = ks$. Numerical analysis of expression (5) shows that, when the distance s is increased, the correlation coefficient decreases rapidly so that when $s > 4\lambda$ the voltages V_1 and V_2 can be considered uncorrelated. In the case of back-scattering the correlation coefficient R(s) does not depend on the form of the expression for the effective diameter of back-scattering and is determined only by the geometry of the problem.

Back-scattering was observed and investigated with the aid of a transmitter with a power of the order of 50 Kw in the pulse; the duration of the pulse was 50 microsecs and the work-

ing frequency 12 Mc. The transmitting antenna consisted of five wave dipoles at a height $\lambda/4$ above the ground; the area of the antenna array was 25 m x 50 m. Measurements of the altitudes of scattering regions and a determination of the degree of regularity of the occurrence of scattered signals were made in August and September, 1958. Observations lasted two months around the clock, for a period of 20 minutes at the beginning of each hour. It was found that a scattered signal may be observed regularly at any time of the day. Throughout the period of observation the critical frequency of the E-layer of the ionosphere did not exceed 4 Mc, i.e. it was at least three times smaller than the working frequency. The scattered signal had a low level, constantly fluctuated and was easily distinguishable from the irregular meteoric "splashes". The scattered pulse was sometimes strongly eroded; often the signal was observed from two or three regions at once. This shows that scattering occurs not just in a narrow segment but in an extensive region of the ionosphere.

Fig. 2 shows histograms giving quantitative characteristics of the altitudes, at which scattering of radio-waves was observed. In these figures we have plotted along the vertical axis the time in which the scattered signal arrived from a definite interval of altitudes (as a percentage of the total signal time). Fig. 2a refers to the daytime (1100-1400 hrs) and graph 2b refers to a nocturnal period (0000-0300 hrs). Analogous histograms were obtained for other intervals of time. It follows from the experiments of Pineo [6] that USW are scattered most effectively by regions of the ionosphere situated in the interval 75-85 km. Eckersley's research on short waves gives altitude values close to those shown in Fig. 2. Such a difference in the altitudes, obtained for ultrashort and short waves, can be explained, if it is assumed that the mean scale of the inhomogeneities increases with altitude. Thus, the regions which scatter

USW most effectively will be situated lower than the corresponding regions for short waves.

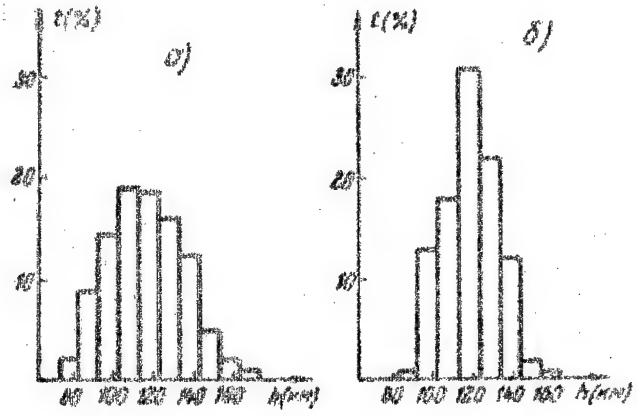

Fig. 2

Formulas (1) and (2) permit us to evaluate the effective diameter of back-scattering. In these experiments the power W of the scattered signal at the receiver input never fell below the noise level (according to our figures $W > 10^{-14}$ watts). Substituting the values of the apparatus parameters in (1) and (2) and assuming that h = 120 km, b = 10 km, we get $\sigma \geq 3.10^{-15}$ m^{-1} for λ = 25 m. On the basis of the above preliminary results it may be stated that the back-scattering of radio-waves by the ionosphere is a regularly observable phenomenon and that the scattering occurs in a region whose thickness is of the order of 10÷20 km, situated at an altitude of 90÷140 km.

Measurement of the magnitude of the scattered signal also makes it possible to determine the effective diameter of back-scattering, which is of great significance in comparing experimental results with the various versions of the theory. An experimental verification of formula (5) is important for corroborating the mechanism of scattering in vertical ionospheric sounding at frequencies above the critical. Moreover, a comparison

between the fluctuational characteristics of the scattered signal and the speed of the wind in the E-region of the ionosphere may help to explain the part played by turbulence in the formation of ionospheric inhomogeneities.

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ON ESTIMATING THE ACCURACY OF AN ADIABATIC INVARIANT

Page 374- 376

by N.G. Denisov

The motion of an electron in a uniform and time-dependent magnetic field is examined. The accuracy of the adiabatic invariant is evaluated on the basis of a single rigorous solution.

When a charged particle moves in a slowly varying magnetic field $\underline{H}$, there exists an adiabatic invariant

$$\mu_m = mv_\perp^2/2H = \text{const} \qquad (1)$$

($v_\perp$ is the component of the electron velocity perpendicular to the direction of the external magnetic field). Various evaluations of the accuracy of this invariant are to be found in the literature. If a small parameter, characterizing the rate of change of the magnetic field, is introduced, then the adiabatic invariant is observed correct to the square of the small parameter [1]. In [2] it was shown that for the particular case of the motion of a particle in a uniform magnetic field, slowly changing with time, the adiabatic invariant is observed with an accuracy proportional to $\exp(-A/a)$ (A = const., a is the small parameter of the problem characterizing the rate of change of the magnetic field).

In the present note it is shown that, using the particular

example examined in [2], these evaluations do not contradict each other but refer to different conditions.

The motion of an electron in a uniform magnetic field is described by the equation (see, for instance, [2]):

$$\ddot{r} + ih\dot{r} + \frac{\dot{h}}{2} hr = 0, \qquad (2)$$

where $h(t) = eH(t)/mc$ is the gyromagnetic frequency of the electron, $r = x + iy$, x and y are coordinates of the electron in the plane perpendicular to the external magnetic field. The origin ($r = 0$) coincides with the axis of symmetry of the problem (the axis of the solenoid). The motion of an electron in a steady magnetic field is described by the solution

$$r = r_0 + \rho e^{-i(ht + \varphi)},$$

i.e. the electron goes through a movement around a circle of radius ρ. The magnetic moment of the current will then equal $\mu_m = err^*/2ch = eh\rho^2/2ch = \text{const}$. In order to investigate (2) in the case of a variable magnetic field $h(t)$, we shall substitute $r = z \exp\left[-\frac{i}{2}\int h dt\right]$ and bring it to the form [2]

$$\ddot{z} + h^2(t) z = 0. \qquad (3)$$

Let us assume that for the interval of time from t_0 to t_1 the function $h^2(t)$ changes from a value $h_0^2(t)$ to $h_1^2(t)$. The solution of equation (3) up to the moment of time t_0 can be written in the form:

$$z = A_1 \exp\left(i\frac{h_0}{2}t\right) + A_2 \exp\left(-i\frac{h_0}{2}t\right), \qquad (4)$$

the coefficient A_1 being put equal equal to zero. This means that initially the electron turns through a circle of radius A_2, the center of which lies on the axis of the solenoid. In such a case solution (4) will describe the motion of an electron in a steady magnetic field. Here it is convenient to introduce the quantity $\mu = (2c/e)\mu_m$.

From (4) we find:

$$\mu_0 = rr^*/h_0 = |A_2|^2 h_0. \qquad (5)$$

In the interval of time (t_0, t_1) the magnetic field varies and when $t = t_1$ it attains the constant value h_1. When $t > t_1$ the solution of (3) is, generally speaking, written in the form:

$$z = c_1 \exp\left(\frac{i h_1 t}{2}\right) + c_2 \exp\left(-\frac{i h_1 t}{2}\right). \qquad (6)$$

Here $c_1 \neq 0$ and, consequently, when $t > t_1$, the electron will turn through a circle of radius c_2, the center of which does not coincide with the axis of the solenoid. The quantity μ will then be equal to

$$\mu_1 = rr^*/h_1 = |c_2|^2 h_1. \qquad (7)$$

The variation may be characterized by the expression

$$D^2 = \mu_0/\mu_1 = |A_2|^2 h_0/|c_2|^2 h_1. \qquad (8)$$

Thus, the calculation of the change in the magnetic moment of a rotating electron for a drop in the magnetic field reduces to finding the relationship between the coefficients of solutions (4) and (6). Similar problems must be solved, for instance, in calculating the reflection of plane waves from inhomogeneous layers. In this case an equation of type (3) describes the propagation of an electromagnetic wave in a layer, the properties of which vary with the coordinate t . The function $h(t)/2$ represents the wave number and μ is proportional to the energy flux carried by the wave. Thus the coefficient D^2 characterizes the leaking of a wave through an inhomogeneous layer. Consequently, a change in μ occurs when, in the analogous problem, there is reflection of waves from the layer.

Let us find the relative change in the magnetic moment

$$(\mu_1 - \mu_0)/\mu_1 = 1 - D^2 = R^2 \qquad (9)$$

(where R^2 is the reflection coefficient), basing ourselves on the solution of known problems on the reflection of waves from an inhomogeneous layer [3]. Let us examine the simplest case of a transitional regime, in which the external magnetic field varies according to the law (transition layer):

$$h^2(t) = [h_1^2 - b^2/(1 + e^{at})]. \qquad (10)$$

The reflection coefficient for such a layer is equal to [3, §73]

$$R^2 = \mathrm{sh}^2\left\{\frac{\pi h_1}{2a}\left(1 - \sqrt{1-p}\right)\right\}\mathrm{sh}^{-2}\left\{\frac{\pi h_1}{2a}\left(1 + \sqrt{1-p}\right)\right\};\ p = b^2/h_1^2 < 1. \qquad (11)$$

Since $h_o^2 = h_1^2 - b^2$, for a small drop in the magnetic field $b^2 = h_1^2 - h_o^2 \approx 2h_1 \Delta h$. Consequently, the parameter $p \approx 2\Delta h/h_1$ characterizes the relative change in the magnetic field. When $p \ll 1$

$$R^2 \approx \mathrm{sh}^2\left(\frac{\pi h_1}{4a} p\right)\mathrm{sh}^{-2}\left(\frac{\pi h_1}{a}\right). \qquad (12)$$

However, if $\pi h_1 p/4a \ll 1$, then $R^2 \approx (\pi h_1 p/4a)^2$. In this case the accuracy of the adiabatic invariant, for a small change in the magnetic field, is determined by the square of the small parameter.

However, if for a given drop in the magnetic field the time of the drop increases ($a \to 0$) and if it can be considered that $\pi h_1 p/4a \gg 1$ ($p \ll 1$), then from (12) we get:

$$R^2 \sim \exp(-2\pi h_1/a) \qquad (p \ll 1).$$

Thus, for a small drop ($p \ll 1$), when $\pi h_1 p/4a = \pi \Delta h/2a \ll 1$, μ is observed correct to $(\pi \Delta h/2a)^2$ and, when $\pi \Delta h/2a \gg 1$, correct to $\exp(-2\pi h_1/a)$. But if the parameter $p = b^2/h_1^2$ is not small, then when $(\pi h_1/2a) \times (1 - \sqrt{1-p}) \gg 1$ the accuracy of the invariant, in accordance with (11), is determined by the factor $\exp\{-(\pi h_1/a)\sqrt{1-p}\}$.

In conclusion, note that analogous evaluations of the

accuracy of the adiabatic invariant could be obtained from a calculation of the coefficient of reflection from inhomogeneous layers in a considerably more general form.

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THE EFFECT OF SPONTANEOUS RADIATION ON THE SPECTRAL LINE WIDTH OF A MOLECULAR GENERATOR

Pages 377-383

by V.S. Troitskiy.

It is shown that the spectral line width of the oscillations of a molecular generator is composed of a width due to the thermal noise of the generator circuit and a width connected with noise resulting from spontaneous molecular radiation. In order of magnitude the latter is equal to the line width of the spontaneous radiation of an isolated molecule in the generator circuit and is the natural threshold of monochromatic oscillation.

In [1] the author found the natural spectral line width of the oscillations of a molecular generator, taking into account thermal noise in the circuit and the shot effect of the molecular flux. This width still does not give a complete representation of the limiting, natural stability of the generator, since it disregards noise due to spontaneous molecular radiation. It therefore seems appropriate to determine, within the framework of the same theory, the effect of spontaneous radiation on fluctuations in the amplitude and frequency of the oscillations.

The question of the spontaneous radiation of a beam of molecules in a resonator in the presence of an external field is at present still without a rigorous quantum-mechanical solution. Nevertheless, on the basis of existing work and given certain simplifying assumptions, we consider it possible to make alternative evaluations of spontaneous radiation in modern generators and its effect on the line width of the generator oscillations. Methodologically, the calculation of fluctuations

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in oscillation is based on a spectral approach [2] and the results obtained in [1], while taking into account the effect of thermal and shot noise, although the action of the latter is introduced in a manner somewhat different from that adopted in [1]. The first section of this article is devoted to the derivation of general expressions connecting the spectral densities of fluctuations in the amplitude and frequency of the oscillations with the intensity and spectrum of circuit and spontaneous radiation noise. The spontaneous radiation is not connected with any of its concrete characteristics or the genetic mechanism (coherence and so on). The second part contains a short review of the conditions, under which spontaneous radiation noise occurs in generators, and, on the basis of an evaluation of this noise, gives an expression for the corresponding line width of the oscillations. Such an evaluation may prove useful in the experimental investigation of the line width of a molecular generator.

1. General expressions for fluctuations in frequency and amplitude.

We shall consider a generator coupled with the load across a circulator. In order to take into account the noise due to spontaneous radiation we shall make it correspond with a certain equivalent emf $\xi_{cn}(t)$, introduced into the generator circuit and having a mean square spectral density $w_{cn}(\omega)$. Since the spontaneous radiation noise is determined by the number of excited molecules, still not undergoing induced emission, it is natural that the emf $\xi_{cn}(t)$ and its spectral density should depend on the amplitude of oscillation of the generator a_o (i.e. $\xi_{cn} = \xi_{cn}(t,a_o)$ and $w_{cn}(\omega,a_o)$).

It is obvious that the width of the spectrum of spontaneous molecular radiation in the generator will not be less than $1/\pi\tau_o$, where τ_o is the most probable transit time for molecules in the circuit. Consequently, the frequency of the oscillations ω_1 will lie within the limits of the spectrum of this noise and the spontaneous radiation noise affects the line width.

The equation for the oscillations of the generator, taking into account spontaneous radiation noise, is:

$$\frac{d^2(V\varepsilon)}{dt^2} + \frac{\omega_0}{Q_n}\frac{d(V\varepsilon)}{dt} + V\omega_0^2 = \omega_0^2(\xi_K + \xi_{cn}), \qquad (1)$$

where $V = a_0 \cos \omega_1 t$ is the voltage in the circuit, ω_0 is the natural frequency of the empty circuit, ω_1 is the average frequency of the steady oscillations, $\xi_K(t)$ is the emf of the thermal noise due to the total resistance of the loaded circuit, Q_H is the Q-factor of the loaded circuit corresponding to this resistance, $\varepsilon = \varepsilon' - i\varepsilon''$ is the dielectric constant of the gas, depending on the amplitude and frequency of the oscillations close to the transition frequency $_2$.

Let the natural resistance of the circuit losses be equal to r_0 and the resistance introduced by the load equal to r_2. To the resistance r_0 there corresponds the Q-factor of the unloaded circuit Q_0 and to the resistance r_2 a Q-factor Q_2 attributable only to the load. The Q-factor of the loaded circuit, corresponding to the total resistance of the circuit $r = r_0 + r_2$, is equal to $Q_H^{-1} = Q_0^{-1} + Q_2^{-1}$. Since the load is coupled with the generator across a circulator, the emf $\xi_K(t)$ of the thermal noise in the circuit will be formed only by its self-resistance r_0, i.e. the spectral density is equal to

$$w_K(\omega) = 4r_0\Theta(T_K, \omega); \quad \Theta(T_K, \omega) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp\frac{\hbar\omega}{kT_K} - 1},$$

where T_K is the temperature of the material of the circuit.

The value of the dielectric constant ε is a function of the number of molecules in the circuit and of the law governing their distribution according to velocity. The latter may be considered independent of time and, consequently, ε undergoes fluctuations connected with the shot effect of the molecular flux. Thus, $\varepsilon = \overline{\varepsilon} + \Delta\varepsilon(t)$; then, assuming $\varepsilon'' = \overline{\varepsilon}'' + \Delta\varepsilon(t)$ and taking into account that $\varepsilon' = 1 + u\varepsilon''$, where $u = (\omega_2 - \omega_1)\tau_0$, we get:

$$\Delta\varepsilon'(t) = u\Delta\varepsilon''(t);$$

$$\Delta\varepsilon(t) = \Delta\varepsilon''(t)(u-1); \quad |\bar{\varepsilon}|^2 = |\bar{\varepsilon}|^2 + 2\bar{\varepsilon}'\Delta\varepsilon'(t) + 2\bar{\varepsilon}''\Delta\varepsilon''(t).$$

Substituting these expressions in equation (1), transfering small fluctuational terms to the right-hand side and neglecting quantities of the second order of smallness, we get (assuming that $iV = \dot{V}/\omega$):

$$\ddot{V} + \left(\frac{\omega_0}{Q_н} + \frac{\omega_0^2}{\omega}\frac{\bar{\varepsilon}''}{|\bar{\varepsilon}|^2}\right)\dot{V} + \omega_0^2\frac{\bar{\varepsilon}'}{|\bar{\varepsilon}|^2}V = \omega_0^2\xi(t,a,V). \qquad (2)$$

where for brevity we have written $\xi(t,a,V) = \xi_K(t) + \xi_{сп}(t,a) + \xi_{Dp}(t,V)$. The emf of the "shot" noise is equal to:

$$\xi_{др}(t,V) = \Delta\varepsilon''(t)(uV - \dot{V}\omega_1). \qquad (3)$$

Let the perturbation solution be $V = a\cos\psi$, where $\psi = \omega_0 t + \vartheta_1 + \varphi(t)$ and $a = a_0(1+\alpha)$, $\alpha(t)$, $\varphi(t)$ being fluctuations; then (see, for example, [2]):

$$\frac{da}{dt} = A_1(a,\omega) - \frac{\omega_0}{2\pi}\int_0^{2\pi}\xi(t,a_0,a_0\cos\psi)\sin\psi\,d\psi;$$

$$\frac{d\psi}{dt} = \omega_0 + B_1(a,\omega) - \frac{\omega_0}{2\pi}\int_0^{2\pi}\xi(t,a_0,a_0\cos\psi)\cos\psi\,d\psi. \qquad (4)$$

Substituting in (4) $a = a_0 + \alpha a_0$ and $\omega = d\psi/dt = \omega_0 + \dot{\vartheta}_1 + \nu(t)$, where $\dot{\vartheta}_1$ is a dynamic frequency correction and $\nu(t) = \dot{\varphi}(t)$ are frequency fluctuations, after expanding in series we get:

$$\frac{d\alpha}{dt} = p\alpha + \frac{p_1}{a_0}\nu(t) - \frac{\omega_0}{2\pi a_0}\int_0^{2\pi}\xi(t,a_0,a_0\cos\omega_1 t)\sin\omega_1 t\,d(\omega_1 t);$$

$$\nu(t) = \frac{qa_0}{1-q_1}\alpha - \frac{\omega_0}{2\pi a_0(1-q_1)}\int_0^{2\pi}\xi(t,a_0,a_0\cos\omega_1 t)\cos\omega_1 t\,d(\omega_1 t), \qquad (5)$$

where $p = (\partial A_1/\partial a)_{a_0}$; $p_1 = (\partial A_1/\partial \omega)\omega_1$; $q = (\partial B_1/\partial a)_{a_0}$; $q_1 = (\partial B_1/\partial \omega)_{\omega_1}$. Substituting its value for $\xi(t, a_0, a_0 \cos \omega_1 t)$ and integrating [2], we get:

$$\frac{d\alpha}{dt} = p\alpha + \frac{p_1}{a_0} v(t) - \frac{\omega_0}{2} \Delta\varepsilon''(t) - \frac{\omega_0}{a_0} \xi(t, a_0) \sin \omega_1 t;$$

$$v(t) = \frac{qa_0}{1-q_1} \alpha - \frac{u\omega_0 \Delta\varepsilon''(t)}{2(1-q_1)} - \frac{\omega_0}{a_0(1-q_1)} \xi(t, a_0) \cos \omega_1 t, \qquad (6)$$

where $\xi(t, a_0) = \xi_K(t) + \xi_{cn}(t, a_0)$. Writing $(\omega_0/a_0)\xi(t, a_0)\sin \omega_1 t = \xi_\alpha$, $(\omega_0/a_0)\xi(t, a_0)\cos \omega_1 t = \xi_v$ and separating the variables in (6), we arrive at the equations:

$$\frac{d\alpha}{dt} = p_0 \alpha - \frac{\omega_0 \Delta\varepsilon''}{2}\left(1 + \frac{p_1 \xi_v}{a_0(1-q_1)}\right) - \frac{p_1 \xi_v}{a_0(1-q_1)} \xi_\alpha;$$

$$\frac{dv}{dt} = p_0 \alpha + \frac{\omega_0 \Delta\varepsilon''}{2(1-q_1)}(pu - qa_0) - \frac{\omega_0 u}{2(1-q_1)} \frac{d\Delta\varepsilon''}{dt} +$$

$$+ \frac{1}{1-q_1}\left\{p\xi_v - qa_0\xi_\alpha - \frac{d\xi_v}{dt}\right\},$$

in which $p_0 = p + p_1 q/(1 - q_1)$.

It is easy to see that $pu - qa_0 \equiv 0$, i.e. the shot effect of the molecular flux does not affect the fluctuations in frequency [1]*. The expressions obtained coincide with expressions given in [1], if the shot terms, which differ in form, are somewhat transformed.

By solving the equation for the spectral components, we get the spectral densities of the fluctuations in amplitude and frequency:

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* This will occur if the absorption line itself is not displaced with respect to frequency in dependence on the number of molecules. In reality, this displacement and change in the shape of the line do exist [3]; the displacement may also be due to variation in the sorting system voltage. The effect of these factors on frequency fluctuation may be taken into account by introducing corresponding fluctuations in ω_2, using, for example, experimental curves for the dependence of the frequency of the oscillations on the gas pressure at the source and the sorting system voltage.

$$w_x(\Omega)=\frac{1}{p_0^2+\Omega^2}\left\{w_\xi(\Omega)\frac{\omega_0^2}{4}\left(1+\frac{p_1 u}{a_0(1-q_1)}\right)^2+\right.$$
$$\left.+\frac{\omega_0^2}{4a_0^2}\left(1+\frac{p_1^2}{a_0^2(1-q_1)^2}\right)\left(w_\xi(\omega_1-\Omega)+w_\xi(\omega_1+\Omega)\right)\right\};$$
$$w_y(\Omega)=\frac{\omega_0^2}{4a_0^2(1-q_1)^2}\left\{w_\xi(\Omega)\frac{u^2a_0^2\Omega^2}{p_0^2+\Omega^2}+\right. \tag{8}$$
$$\left.+\frac{p^2+q^2a_0^2+\Omega^2}{p_0^2+\Omega^2}\left(w_\xi(\omega_1+\Omega)+w_\xi(\omega_1-\Omega)\right)+\frac{2qa_0\Omega}{p^2+\Omega^2}\left(w_\xi(\omega_1-\Omega)-w_\xi(\omega_1+\Omega)\right)\right\},$$

where

$$w_\xi(\omega_1\pm\Omega)=w_x(\omega_1\pm\Omega)+w_{cn}(\omega_1\pm\Omega).$$

If we take into account that the expressions obtained hold true for $\Omega < 2/\tau_0$ [1] and that we are interested in the spectrum $w_y(\Omega)$ for the very low frequencies Ω, defining the spectral line width of the oscillations [2], then in the frequency range under consideration $\Omega \simeq 0$, even for spontaneous radiation noise, it is sufficiently accurate to put $w_{cn}(\omega_1-\Omega)+w_{cn}(\omega_1+\Omega)=2w_{cn}(\omega_1)$. A similar equation for thermal noise will be accurate when $\hbar\omega \ll kT_K$ and approximate when $\hbar\omega \gg kT_K$. Hence it follows that $w_\xi(\omega_1-\Omega)-w_\xi(\omega_1+\Omega)\simeq 0$ and the correlation term of (8) is eliminated* [2]. Finally, since the components of the spectrum w_y, proportional to Ω^n when $n \leqslant 2$, do not affect the line width and, moreover, $\Omega \ll p \simeq p_0$, these terms may be neglected.

Substituting values of the coefficients p,q, presented in [1], and taking into account that $a_0^2 = P_\sim Q_H 2r$, where $P_\sim$ is the power of the oscillations, we get:

$$w_x=\frac{\eta_1^2}{(\eta_1-1-u^2)^2}\frac{Q_H^2w_c(\Omega)}{4}+\frac{(\eta_1-2u^2)^2+4u^2}{(\eta_1-1-u^2)^2}\frac{w_\xi(\omega_1)+w_{cn}(\omega_1,P_-)}{4rP_-};$$
$$w_y=\frac{\omega_0^2}{Q_H^2}\frac{w_\xi(\omega_1)+w_{cn}(\omega_1,P_-)}{4rP_-}(1+u^2). \tag{9}$$

* In this term in [1] the spectral densities $w_\xi(\omega_1-\Omega)$ and $w_\xi(\omega_1+\Omega)$ are wrongly added; however, the whole term is also discarded on account of its being small and vanishing when $\Omega = 0$.

Here $\eta = (8\pi\hbar/Sl)\beta N \tau_0^2 Q_H$ is the excitation factor, S is the cross section of the molecular flux and the area of the circuit capacitor (space factor equal to unity), l is the length of flux in the capacitor (circuit), $Q_2 = 0.5\omega_2\tau_0$ is the "Q-factor" of the absorption line of the gas and N is the number of molecules coming into the resonator per second. It is evident that in (9) the quantity $\varkappa r_K/r$ is equal to $4\theta(T_K, \omega_1)r_0/(r_0 + r_2)$ (if the circulator were not there, then $\varkappa r_K/r = 4\theta(T_K, \omega_1)$); the quantity $\varkappa r_{cn}(\omega_1, P_\sim)/r$ is equal to the spectral power density of the spontaneous radiation in the resonator $P_{cn}(\omega_1, P_\sim)$ at the oscillation frequency $\Delta\varepsilon''(t) = \bar{\varepsilon}''m(t)/M$, where $m(t) = M(t) - \bar{M}$ is the fluctuation in the number of molecules M(t) in the generator circuit. In accordance with [1], the spectrum m(t) is equal to $\varkappa_m \approx 12N\tau_0^2$, so that

$$\varkappa_\varepsilon(\Omega) = 3/Q_H^2 N.$$

Taking into account the above reasoning, we shall have:

$$w_0 = \frac{\eta^2}{(\eta - 1 - u^2)^2}\frac{3}{4N} + \frac{(\eta - 2u^2)^2 + 4u^2}{(\eta - 1 - u^2)^2}\left[\frac{Q_K}{Q_0}\frac{\Theta(T_K, \omega_1)}{P_\sim} + \frac{P_{cn}(\omega_1, P_\sim)}{P_\sim}\right]. \quad (10)$$

The power of the oscillations is determined from the condition $\beta\tau_0 E_0^2 = \eta - 1 - u^2$, where E_0 is the amplitude of the intensity of the field in the circuit capacitor, equal to a_0/l. Expressing E_0 in terms of $P_\sim$, we find that $P_\sim = N\hbar\omega_2\eta^{-2}(\eta - 1 - u^2)$. In the limit, for strong excitation ($\eta \gg 1$), the power is equal to $N\hbar\omega_2$. Assuming for the sake of simplicity that the generator is sufficiently accurately tuned ($u = 0$ and $\omega_1 = \omega_2$), we finally get:

$$w_0 = \frac{3}{4N} + \frac{\Theta(T_K, \omega_2)}{N\hbar\omega_2}\frac{Q_K}{Q_0} + \frac{P_{cn}(\omega_2, P_\sim)}{4N\hbar\omega_2};$$

$$w_1 = \frac{\Theta(T_K, \omega_2)}{\hbar\omega_2}\frac{Q_K}{Q_0}\frac{4}{N\tau_0} + \frac{P_{cn}(\omega_2, P_\sim)}{\tau_0^2 N\hbar\omega_2}. \quad (11)$$

The spectral line width is equal to:

$$\Delta F = \frac{\Theta(T_к, \omega_2)}{\hbar\,\omega_2}\,\frac{Q_н}{Q_0\,\pi N \tau_0^2} + \frac{P_{сп}(\omega_2, P_\sim)}{\hbar\,\omega_2}\,\frac{1}{4\pi N \tau_0^2}. \qquad (12)$$

Note that, if there is no circulator, the ratio $Q_н/Q_0$ should be replaced with unity.

2. Quantitative evaluation

Since there is still no accurate theory of spontaneous radiation noise in a working generator, we can only make certain estimates of the spectral line width of the generator, connected with spontaneous radiation. Nevertheless, it is still possible to make certain general conclusions about the dependence of this width on the parameters. Since the width of the spontaneous radiation noise spectrum cannot be less than $1/\pi\tau_0$, it is possible to show that, with an accuracy sufficient for purposes of estimation,

$$P_{сп}(\omega) \simeq \frac{2\tau_0 P_{сп}}{1+(\omega_2-\omega)^2\tau_0^2}. \qquad (13)$$

This expression satisfies the normalization condition $\int_0^{2\pi} P_{сп}(\omega)\,df = P_{сп}$, where $P_{сп}$ is the total power of spontaneous radiation in the resonator.

Without limiting the generality of the reasoning, it is possible to assume that $P_{сп} = AM\Gamma_0\hbar\omega_2$, where Γ_0 is the probability of radiation from an isolated molecule in the circuit, M is the total number of excited molecules in the circuit, which there would be in the absence of generation, and A is a certain factor depending on, generally speaking, the power of the oscillations, the Q-factor of the circuit and, perhaps, on other generator parameters, e.g. the number of molecules and so on. Such an expression denotes that the actual power of the spontaneous radiation is expressed in terms of the power of the incoherent spontaneous radiation $M\Gamma_0\hbar\omega_2$, which the molecules in the generator circuit would have in the absence of generation and if their acts of emission were independent.

To a certain extent the appropriateness of such a formulation may be based, for example, on the following considerations. In the molecular beams ordinarily used in generators the distance between molecules is much less than the wavelength. As shown in [4], in this case the radiation from individual molecules, concentrated in a volume of the order of λ^3, is interdependent. The whole cluster of molecules may be in a state of coherent spontaneous radiation, the power and spectrum width of which differ from the corresponding values for independent molecular radiation.

The intensity and spectrum of the coherent spontaneous radiation of the cluster are defined by a certain cooperative quantum number r for the whole molecular system and a number $m = (M_+ - M_-)/2$, equal to half the excess of excited molecules in the cluster concerned. In the section of the molecular beam, lying in the generator circuit, $r = m = M/2$ (assuming that the volume is of the order of or less than λ^3). In free space the intensity of coherent spontaneous radiation from such a cluster is equal to [4]:

$$P_{cn} = \gamma_0 \hbar \omega_2 (r - m + 1)(r + m) = M \gamma_0 \hbar \omega_2, \qquad (14)$$

where γ_0 is the probability of spontaneous radiation from an isolated molecule in free space. As is known, $\gamma_0 = 4\omega_2^3 d^2/3\hbar c^3$, where d is the dipole moment of the transition. For ammonia radiation at $\lambda = 1.25$ cm $\gamma_0 = 1.5 \cdot 10^{-7}$ sec. It is evident from (14) that the power of coherent spontaneous radiation coincides with the power which would be observed in independent molecular radiation. The most intense radiation will occur when $m = 0$, $r = M/2$, which gives $P_{cn} = \frac{1}{4} M^2 \gamma_0 \hbar \omega_2$. However, the width of the spontaneous radiation spectrum will be $\Delta f_{cn} = r\gamma_0/2\pi$ (see, for example, [5]), i.e. r times greater than the width of the incoherent spontaneous radiation spectrum $\gamma_0/2\pi$. For ordinary beams $r = M/2 \simeq 0.5 \cdot 10^9$ and $\Delta f_{cn} \simeq 12$ cps. However, in a generator the molecular beam is in a circuit with losses. Obviously, expression (14) can be used, if we take into account the fact that the

circuit only alters the probability of spontaneous radiation for a molecule lying in it. As shown in [6], the probability of spontaneous radiation from a single isolated molecule in the circuit (i.e. in the absence of other molecules) is equal to $\Gamma_0 = \gamma_0 Q_H \lambda^3/6\pi^2 v_K$, where v_K is the circuit volume. Thus, in the absence of oscillations the total power of the spontaneous radiation in the generator circuit is obviously close to $P_{cn} = M \Gamma_0 \hbar \omega_2$.

It should be noted that an analogous expression has been obtained by Pound [7], for a two-level equilibrium system of molecules in a circuit. The treatment was thermodynamic; a quantum-mechanical expression was used only for the dielectric constant. This author obtains a certain quantity $\Gamma_0 = \gamma_0 3Q_H \lambda^3/4\pi^2 v_K$, which should be interpreted as the probability of radiation from a single molecule in a system of molecules in a resonator. For ordinary conditions $\Gamma_0 \approx \gamma_0 Q/50 \approx 100\,\gamma_0$. In generation the conditions of spontaneous radiation change sharply and it is natural to write finally

$$P_{cn} = AM\Gamma_0 \hbar \omega_2 = 2A\tau_0 \Gamma_0 P_- . \qquad (15)$$

Here the condition of smallness of the spontaneous radiation noise $2A\tau_0\Gamma_0 \ll 1$, on which the calculation of the fluctuations in oscillation is based, is assumed to be fulfilled.

It is evident that in the absence of generation A is close to unity and that in generation it may obviously be considerably greater than unity depending on the regime. In fact, on considering the motion of a certain part of the beam of molecules in the resonator, it is possible to see that, thanks to induced emission, in a known time the quantity m becomes equal to zero and there ensues a state of coherent radiation, the power of which is proportional to the square of the number of molecules in the part of the beam concerned.

Taking into account (13), (15), we get, in accordance with (12),

$$\Delta F = \Delta F_1 + \Delta F_{cn} = \frac{\Theta(T_K, \omega_2)}{\hbar\omega_2} \frac{Q_H}{Q_0 \pi N_0^2} + \frac{A\Gamma_0}{\pi} . \qquad (16)$$

If we take $\Gamma_0 \simeq 100\gamma_0 = 1.5 \cdot 10^{-5}$, then $\Delta F_{cn} = 0.5 \cdot 10^{-5}$ cps. In the latter expression we shall reckon that the absorption line width $\Delta f_l = 1/\pi\tau_0$ and $2N\tau_0 = M$. Then we shall get the interesting expression:

$$\Delta F = 2\frac{\Theta(T_K, \omega_2)}{\hbar\omega_2}\frac{\Delta f_l}{M}\frac{Q_H}{Q_0} + \frac{A\Gamma_0}{\pi}. \qquad (17)$$

When $\Delta f_l = 3 \cdot 10^3$ cps, $Q_H/Q_0 \simeq 0.05$, $M = 10^{10}$, $T_K = 300°K$ the "thermal" line width is equal to $\Delta F_T \simeq 10^{-5}$ cps, i.e. it becomes comparable with the line width due to spontaneous radiation. The "thermal" line width may be reduced still more by cooling the circuit and a rational choice of the ratio Q_H/Q_0. When $\hbar\omega_2 > kT_K$, then $\Theta(T_K, \omega_2) \simeq \hbar\omega_2/2$ and

$$\Delta F = \frac{Q_H}{Q_0}\frac{\Delta f_l}{M} + \frac{A\Gamma_0}{\pi}. \qquad (18)$$

Thus, the minimum limiting line width of the molecular generator is equal in order of magnitude to the line width of the radiation of a molecule in the resonator.

The expressions obtained in this article can, we are convinced, serve as the basis for the experimental investigation of spontaneous radiation noise in a generator by measuring the spectral line width of its oscillations.

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ON TECHNIQUES IN ATOMIC BEAM RADIOSPECTROSCOPY*

Pages 384-387

by L.L. Myasnikov

Starting from semi-classical theory all the fundamental results of the theory of the atomic beam magnetic resonance method are given by means of simple evaluations.

The atomic beam magnetic resonance method with separate variable fields, proposed by Ramsey [1], like Rabi's method (see for example [2]), can be used to determine spectral absorption lines lying in the radio range. These lines may correspond to transitions between atomic magnetic levels in weak magnetic fields, between levels governed by quadrupole interactions, spin levels of molecules, nuclear and molecular magnetic levels in powerful magnetic fields and so on. Limiting ourselves to the case of beams of atoms of the alkali elements, we shall consider certain conclusions of the simplified theory.

In the schematic diagram of Fig. 1 an atomic beam, emerging from the oven П, passes in a vacuum through the sections A and B, in which there are formed inhomogeneous magnetic fields, serving to sort the atoms with respect to state (in dependence on the sign of the effective dipole magnetic moment) and to deflect the sorted beams. Between A and B there is a section C with a uniform steady magnetic field, at the beginning and end of which lie the resonators E_1 and E_2, excited by the SHF generator (CB4) at the resonance frequency.

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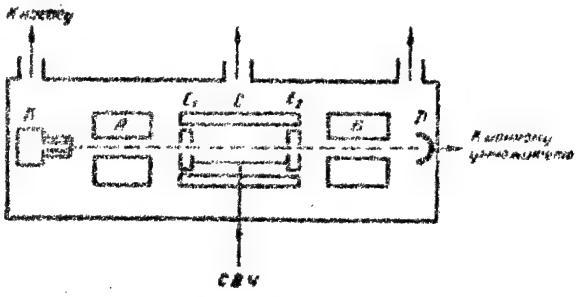

Fig. 1

For atoms with a quantum number for the electron moment of momentum equal to $J = \pm 1/2$ (the alkali elements, such as cesium, belong here) in resonance transitions induced by a radiofrequency field the sign of the effective dipole moment μ changes and, accordingly, there is a corresponding change in the direction of the force $\mu \partial B/\partial z$ (B is the magnetic induction), deflecting the atom in the inhomogeneous magnetic field. In the case of uniform orientation of the inhomogeneous magnetic fields it is possible to reduce to a minimum the intensity of the beam, passing through the system described in the absence of a variable field and impinging on the ion detector D . Thanks to the transitions to the second state, induced by the variable field (absorption), the intensity of the atomic beam falling on the detector will be increased. The "zero method", based on this, is analogous to Tepler's optical method, in which the optical system is set up in darkness with a Foucault knife-edge, so that when the refractive index changes there is a redistribution of intensity giving bands. Fig. 2 shows the path of an atomic beam in the inhomogeneous magnetic fields (the broken line indicates transition to the second state); the analogous optical system is shown below. The role of the Foucault knife-edge is played by an inhomogeneous magnetic field in conjunction with sliding diaphragms. It is possible to use the classical analogy with "refraction" by introducing a certain equivalent refractive index, characterizing the deflection of beams, in which transitions have

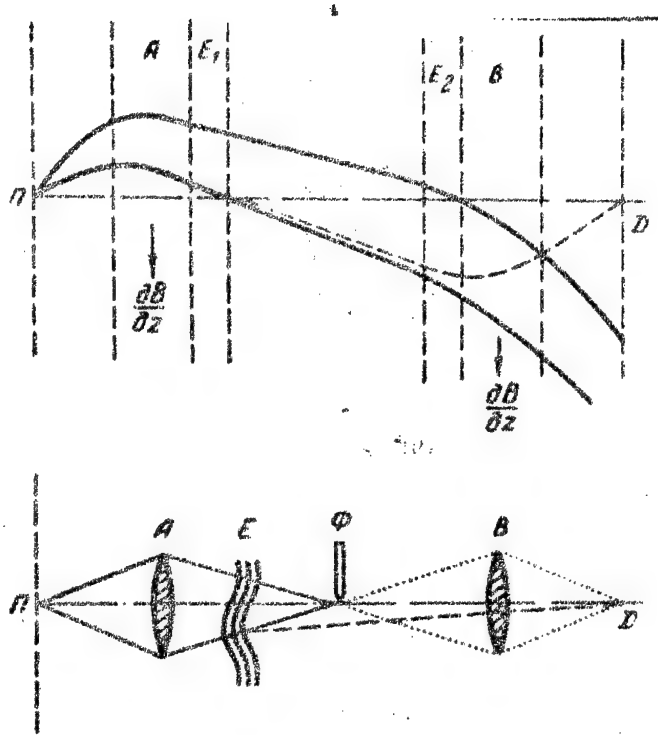

Fig. 2

For such transitions resonance must occur; the frequency of the variable field in the resonators must coincide with the Larmor frequency:

$$\omega_0 = (W_2 - W_1)/\hbar = \gamma_J H_0,$$

where γ_J is the gyromagnetic ratio, $\hbar$ is Planck's constant, 1 and 2 are the numbers of the states and H_0 is the strength of the steady magnetic field.

Preserving the framework of semi-classical theory, we shall introduce the notion of the orientational wavelength for an atom traveling with a velocity α in a space containing a uniform steady and a variable field. This wavelength

$$\lambda = 2\pi\alpha/\omega_{op}$$

where ω_{op} is the reorientation frequency of the effective dipole moment, equal to $\omega_{op} = \gamma_J H_1$ (H_1 is the amplitude of the variable magnetic field). If we introduce the Larmor frequency, then

$$\omega_{op} = \omega_o H_1/H_o.$$

In the interval with variable fields, having a total length 2ℓ, there should be a "flipping" of the dipole moment*, i.e. a transition of the atom to the second state, dependent on the original background given by the scattered atoms falling on the ion detector. This transition disturbs the minimum setup. It is therefore necessary that

$$2\ell \sim \Lambda/2$$

This optimum condition leads to the expression

$$\omega_o (H_1/H_o)\ell/\alpha = \pi/2, \qquad (1)$$

which practically coincides with the condition given by Ramsey

$$2b \frac{\ell}{\alpha} = 0.6\pi.$$

Here α is the most probable velocity of the atom and

$$b = \hbar^{-1} e^{i\omega_o t} (2 |V| 1),$$

where $(2 |V| 1)$ is the matrix element of the interaction energy, corresponding to a transition from the state 1 ($m_J = -1/2$) to the state 2 ($m_J = +1/2$). The quantity b is equal to

$$b = \omega_o H_1/2H_o.$$

The natural width of the resonance line of induced absorption flows from the classical uncertainty relation

$$\Delta\nu\, t_\alpha \sim 1$$

where $\Delta\nu$ is the width of the frequency spectrum of a pulse of duration t_α, t_α being the transit time for an atom. The spectrum width $\Delta\nu$ may be expressed by means of the half line width $\Delta\nu_o$, measured at the half-intensity level:

$$2\Delta\nu_o \sim 1/2t_\alpha .$$

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* Roughly speaking, over the length E_1 the phase of the orientational oscillations of the moment will change by $\pi/2$ and after E_2 again by $\pi/2$.

In the radiospectroscopic method under consideration the intensity distribution has an interference character. The interference of orientational waves, coming from E_1 and E_2, is important in this connection; the spectrum is found from expanding as a Fourier integral a pulse, the duration of which is equal to the transit time between resonators. Thus, $t_\alpha \sim L/\alpha$ and the width of the maximum is

$$2\Delta\nu_0 \sim 0.5\alpha/L. \qquad (2)$$

This practically coincides with the expression $2\Delta\nu_0 = 0.65\alpha/L$, given by Ramsey.

Close to resonance there is observed a sharp interference pattern, giving a few interference bands in the center of the contour of the resonance line. This pattern depends on the shift in the phases of the oscillations in the resonators. The interference patterns obtainable for different phase-shifts can be approximately represented as the result of a shift in the sinusoidal intensity distribution close to resonance for a definite phase angle (in dependence on the mistuning $\Omega = (\omega_0 - \omega)/2\pi = \nu_0 - \nu$ (see Fig. 3)). This effect suggests the interference of beams from two elements of a spectrograph (two slits, two plates and so on). For a phase-shift of $\pi/2$ the band is displaced by $\pi/2$. The "period" of the sinusoid is approximately equal to $4\Delta\nu_0 t_\alpha$. For aphase-shift of the maximum is shifted by $2\Delta\nu_0 \delta t_\alpha/\pi$.

Somewhat further from the center the contour of the resonance line has the Lorentz form, the width being determined by the expression:

$$2\Delta\nu_1 \sim \alpha/l \qquad (3)$$

which coincides with that used in Rabi's method. Thus, on the contour of the absorption line close to the maximum there is superimposed a sinusoidal periodic structure, making possible a considerable increase in the resolving power of the radiospectroscope. This coincidee with results of the detailed theory, worked out by Ramsey.

I would like to express my warm gratitude to V.F. Volkov, A.I. Naumov and L.N. Khodaleva for their part in preparing this article.

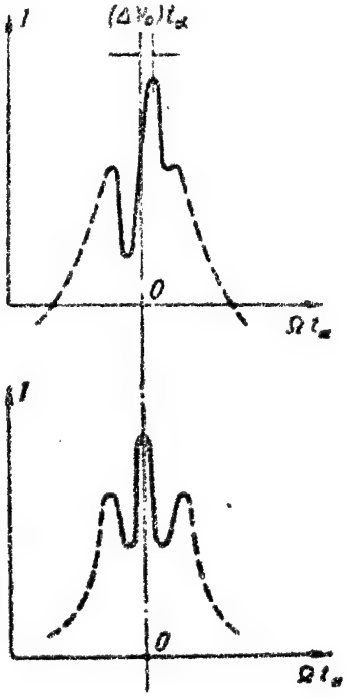

Fig. 3.

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EFFECT OF A BOUNDARY SURFACE ON THE FLUCTUATION OF RADIOWAVES PROPAGATED IN AN INHOMOGENEOUS MEDIUM .

Pages 388-394 by A.V.Men', V.I.Gorbach, S.Ya.Braude.

Approximate expressions are obtained for determining fluctuations in amplitude and phase in the near and far zones for propagation in an inhomogeneous medium above a plane boundary surface. There is shown to be an essential difference in the dependence of the intensity of the fluctuations on range, wavelength and altitude compared with the case of free space. Theoretical calculations and experimental findings are compared.

A number of papers have been published [1-7] on the theory of the propagation of electromagnetic waves in an unbounded turbulent medium. In the general case the prospect of using the results of this work for propagation in an inhomogeneous medium, bounded by a boundary surface, is not obvious. In particular, experimental measurements of fluctuations in SHF radiowaves propagated close to the surface of the earth show that the basic dependences of phase and amplitude fluctuations on range, wavelength and so on differ substantially from those calculated. Moreover, existing theory does not satisfactorily account for a series of effects observed under these conditions.

These considerations lead to the conclusion that a boundary surface has an important effect on fluctuations in electromagnetic waves propagated along it.

1. We shall consider in the approximation of geometrical optics [8] the fluctuations in amplitude and phase of radiowaves, propagated in a turbulent medium in the presence of a plane boundary surface.

In this case the resultant field at the receiving point e will be determined by a direct wave (e_1) and a wave (e_2) reflected from the boundary surface (see Fig. 1):

$$e = e_1 + e_2;$$
$$e_1 = E_1 \exp|-j(\omega t - \psi_1)|; \qquad e_2 = |R| \exp(j\Theta) E_2 \exp|-j(\omega t - \psi_2)|;$$
$$e = E \exp|-j(\omega t - \psi)|. \qquad (1)$$

Here R and Θ are the modulus and phase of the Fresnel reflection coefficient R.

We shall write the signal amplitudes and phases in the following form:

$$E_1 = E_{01} + \Delta E_1(t); \quad E_2 = E_{02} + \Delta E_2(t); \quad E = E_0 + \Delta E(t);$$
$$\psi_1 = \psi_{01} + \Delta\psi_1(t); \quad \psi_2 = \psi_{02} + \Delta\psi_2(t); \quad \psi = \psi_0 + \Delta\psi(t), \qquad (2)$$

where E_{01}, E_{02}, E_0, ψ_{01}, ψ_{02}, ψ_0 are mean values and ΔE_i and $\Delta\psi_i$ are the fluctuational components of the corresponding quantities.

For the case $r_1 \approx r_2$ (Fig. 1), assuming

$$E_{01} \approx E_{02}; \quad |R| = 1; \quad \Theta = \pi \qquad (3)$$

and introducing the notation

$$\psi - \psi_{01} = \varphi; \quad \psi_{02} - \psi_{01} = \varphi_1, \qquad (4)$$

we write (1) in the following form (omitting the time multiplier):

$$E \exp(j\varphi) = (E_{01} + \Delta E_1) \exp|j\Delta\psi_1| - (E_{01} + \Delta E_2) \exp|j(\varphi_1 + \Delta\psi_2)|. \qquad (5)$$

In the case of small fluctuations, when $\Delta E_{1(2)}/E_{01} \ll 1$ и $\Delta\psi_{1(2)} \ll 1$, the mean amplitude $\bar{E} \approx E_0$ and the fluctuations $\Delta E/E_0$ of the total field outside the region of interference minima ($\varphi_1 - 2\pi n \gg \Delta\psi_2 \pm \pm \Delta\psi_1$) may be computed on the basis of the following approximate

expressions:

$$E_0 \approx 2E_{01} \sin(\varphi_1/2); \tag{6}$$

$$\frac{\Delta E}{E_0} \simeq \frac{\Delta E_1 + \Delta E_2}{2E_{01}} + \frac{\Delta\psi_2 - \Delta\psi_1}{2} \operatorname{ctg}(\varphi_1/2). \tag{7}$$

With the same assumptions the phase of the resultant signal is

$$\bar{\varphi} = \psi_\Phi - \psi_{01} \approx -(\pi/2 - \varphi_1/2); \tag{8}$$

$$\Delta\phi = \frac{\Delta E_1 - \Delta E_2}{2E_{01}} \operatorname{ctg}(\varphi_1/2) + \frac{\Delta\psi_1 + \Delta\psi_2}{2}. \tag{9}$$

Passing from random variations in amplitude and phase to mean squares and considering the medium isotropic:

$$\overline{(\Delta E_1/E_{01})^2} = \overline{(\Delta E_2/E_{01})^2}; \qquad \overline{(\Delta\psi_1)^2} = \overline{(\Delta\psi_2)^2}, \tag{9a}$$

we find that

$$\begin{aligned} 2\overline{(\Delta E/E_0)^2} &\approx \overline{(\Delta\psi_1)^2}(1 - R_\psi)\operatorname{ctg}^2(\varphi_1/2) + \overline{(\Delta E_1/E_{01})^2}(1 + R_E); \\ 2\overline{(\Delta\phi)^2} &\approx \overline{(\Delta E_1/E_{01})^2}(1 - R_E)\operatorname{ctg}^2(\varphi_1/2) + \overline{(\Delta\psi_1)^2}(1 + R_\psi), \end{aligned} \tag{10}$$

where R_E and R_ψ are the correlation coefficients of the amplitude and phase fluctuations in the signals:

$$R_E = \overline{\Delta E_1 \Delta E_2}/\overline{(\Delta E_1)^2}; \quad R_\psi = \overline{\Delta\psi_1 \Delta\psi_2}/\overline{(\Delta\psi_1)^2};$$

$$\varphi_1 \approx 4\pi h_1 h_2/r\lambda.$$

Note that in obtaining expressions (10) we neglected the small terms

$$\overline{\Delta\psi_2 \Delta E_2} - \overline{\Delta\psi_1 \Delta E_1}, \quad \overline{\Delta\psi_2 \Delta E_1} - \overline{\Delta\psi_1 \Delta E_2}, \tag{11}$$

which is valid when condition (3) is fulfilled, assuming the medium above the boundary surface to be isotropic [3,4].

On analogy with the average field formulas, obtained on the basis of an interference model [8,9], expressions (10) may be considered "reflection" formulas for the fluctuations. It should be noted that these expressions degenerate into formulas for "free space" when $R_{E(\psi)} \to 1$. (This case occurs, in particular, when the angle of

elevation $\gamma \to \pi/2$ (see Fig. 1)).

In the region of interference minima ($\varphi_1 \approx 2\pi n$) we have the following relationships:

$$E \approx \sqrt{(\Delta E_1 - \Delta E_2)^2 + (\Delta\psi_1 - \Delta\psi_2)^2 E_{01}^2}\ ;$$
$$\operatorname{tg}\psi \approx (\Delta\psi_1 - \Delta\psi_2) E_{01}/(\Delta E_1 - \Delta E_2). \qquad (12)$$

It follows from (12) that in this case the relative variations in amplitude tend to infinity and the phase ψ may assume any values.

In making numerical calculations as a first approximation it is convenient to assume that the fluctuations in the amplitudes and phases of the individual components e_1 and e_2 of the resultant field (9a) are the same as in the absence of a boundary surface. Then for a correlation function of pulsations in the refractive index μ of the troposphere of the form

$$\overline{\mu_1 \mu_2} = \overline{\mu^2} \exp(-\rho^2/l^2), \qquad (13)$$

where l is the scale of the inhomogeneity and ρ is the distance between points in space, it is possible to get [2,5]:

$$\left(\frac{\overline{\Delta E_1}}{E_{01}}\right)^2 \approx \left(\overline{\ln \frac{E_{01} + \Delta E_1}{E_{01}}}\right)^2 = 2\pi^2 \overline{\mu^2} \frac{l r_1}{\lambda^2}\left(1 - \frac{\operatorname{arc tg} D}{D}\right); \qquad (14)$$
$$\overline{(\Delta\psi_1)^2} = 2\pi^2 \overline{\mu^2} \frac{l r_1}{\lambda^2}\left(1 + \frac{\operatorname{arc tg} D}{D}\right);$$
$$D = 2r_1 \lambda/\pi l^2.$$

To determine R_E and R_ψ we turn to Fig. 1. It is possible to show, for example, when $r'_{1(2)} \gg l$ and $r''_{1(2)} \gg l$, that in the far zone, ($D \gg 1$), correct to terms of the order of l/r_1 [5,6]:

$$R_\psi = R_E = \frac{\sqrt{\pi}}{2} \frac{l}{z} \operatorname{erf}\left(\frac{z}{l}\right) = R\left(\frac{z}{l}\right). \qquad (15)$$

where

$$z = \frac{2h_1 h_2}{h_1 + h_2} \frac{1}{\sqrt{1 + (h_2 - h_1)^2 r_0^{-2}}}.$$

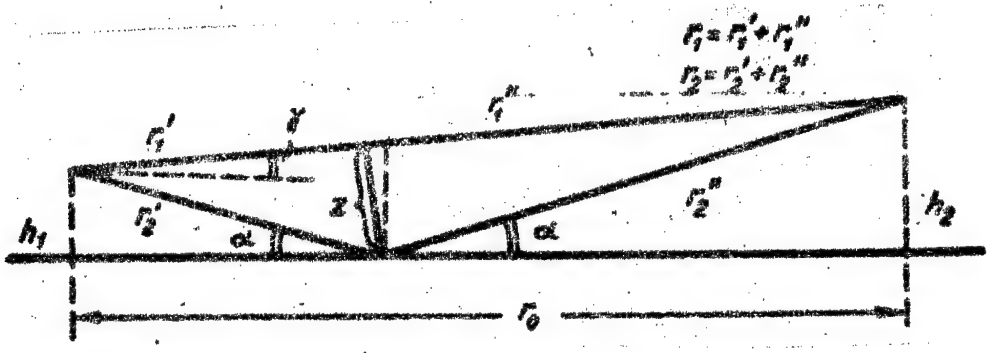

Fig. 1: Geometry of the communication line.

In this case, in accordance with (14)

$$\left(\frac{\overline{\Delta E_1}}{E_{01}}\right)^2 = (\overline{\Delta\varphi_1})^2 = 2\pi^{7/6}\overline{\mu^2}\frac{l r_1}{\lambda^2}. \qquad (14')$$

Then expression (10) takes the form:

$$\left(\frac{\overline{\Delta E}}{E_0}\right)^2 = (\overline{\Delta\varphi})^2 = \pi^{7/6}\overline{\mu^2}\frac{l r_1}{\lambda^2}\left[\left(1-\frac{\sqrt{\pi}}{2}\frac{l}{z}\operatorname{erf}\frac{z}{l}\right)\operatorname{ctg}^2\left(\frac{\varphi_1}{2}\right)+ 1+\frac{\sqrt{\pi}}{2}\frac{l}{z}\operatorname{erf}\frac{z}{l}\right], \qquad (16)$$

i.e. within the framework of the assumptions made for propagation above a boundary surface, as for an unbounded turbulent medium, in the far zone the fluctuations in amplitude and phase of the resultant signal are equal. However, the dependence of the intensity of the fluctuations on the wavelength, the parameter l and the range r_1 may then prove to be substantially different than for free space.

In the region of small angles of elevation, where Vvedenskiy's known quadratic formula ($E_0 \sim 1/r_1^2$) holds for the average field

$$\operatorname{ctg}(\varphi_1/2) \approx (2/\varphi_1),$$

and when

$$|1 - R(z/l)|\,(r_1\lambda/2\pi h_1 h_2)^2 \gg 1 + R(z/l)$$

expression (16) is converted to the form:

$$\overline{(\Delta\psi)^2} = \left(\frac{\Delta E}{E_0}\right)^2 \approx \frac{\sqrt{\pi}}{4}\mu^2 \frac{l r_1^3}{h_1^2 h_2^2}\left[1 - \frac{\sqrt{\pi}}{2}\frac{l}{z}\operatorname{erf}\frac{z}{l}\right]. \qquad (16a)$$

Thus, as follows from (16) and (16a), in dependence on the correlation of fluctuations in the direct and reflected waves the square-law decrease in the average field with distance in the region of the lower lobe may be accompanied by an increase in the intensity of the fluctuations proportional to r_1^a, where $1 \leqslant a \leqslant 3$.

2. To illustrate the relationships obtained, Fig. 2 shows calculated dependences of the intensity of phase fluctuations on the range in the region of the lower lobe for different values of the parameter l when h_1 = 5 m, h_2 = 10 m and λ = 10 cm. These dependences are normalized with respect to the value of the fluctuations when r_1 = 10 km. Experimental data, obtained from three experiments over the sea, are also plotted for the sake of comparison.

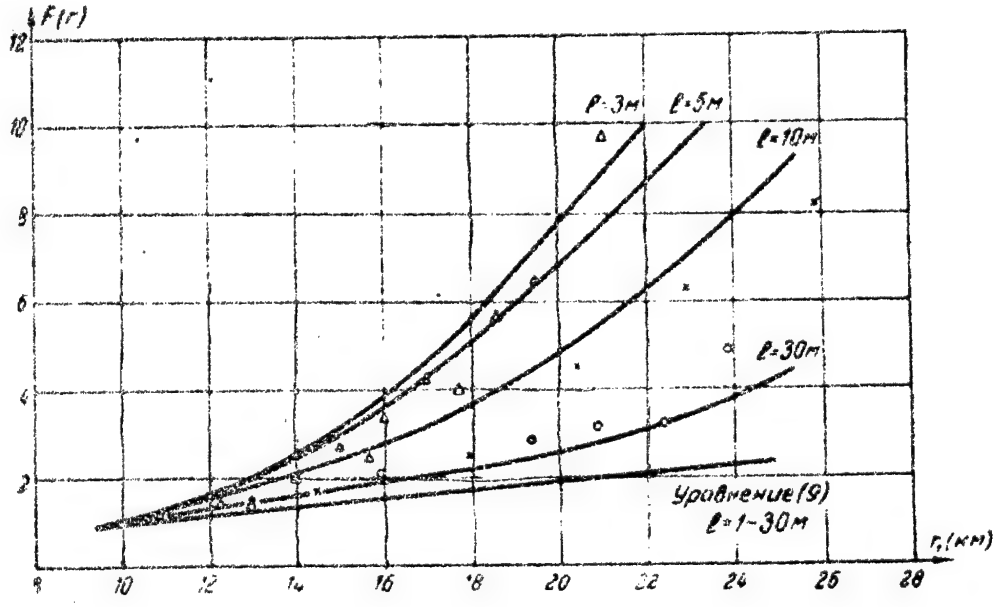

Fig. 2: Intensity of fluctuations as a function of the range $F(r_1) = \overline{(\Delta\psi)^2_{r_1}}/\overline{(\Delta\psi)^2_{r_1=10\,km}}$ (h_1 = 5 m; h_2 = 10 m; λ = 10 cm; X, O, Δ – experimental data.)

In the troposphere the parameter l, an averaged characteristic of the turbulent processes, varies from experiment to experiment within rather wide limits, as indicated by measurements [10].* From this it follows that the relative dependence of the intensity of the fluctuations on range also ought not to remain constant, as was in fact observed in measurements made over the sea.

As we noted, taking into account the effect of the boundary surface may also lead to a substantial change in the frequency dependence of the intensity of the fluctuations compared with the case of free space. Fig. 3 shows the calculated dependence of $(\overline{\Delta\varphi})^2$ on the wavelength in the range λ = 1 - 100 cm for the region of the lower lobe for h_1 = 5 m, h_2 = 40 m and r_1 = 30 km. As follows from the figure, in the range mentioned there is always a tendency towards a weakening of the frequency dependence compared with the relation (14) proportional to $1/\lambda^2$. This is due to the fact that the increase in the fluctuations of the components for a decrease in the wave proportional to $1/\lambda^2$ is compensated by a change in the angle of displacement φ_1 between them ("sinking" of the lobe).

As follows from Fig. 3, for a fixed route the frequency dependence may vary substantially from experiment to experiment in connection with changes in the parameter l. As l increases, this dependence approaches the case of free space (equation (14)).

We shall now consider the "height" dependence of the fluctuations. Fig. 4 shows the calculated dependence of the fluctuations on the quantity h_2 for different l when h_1 = 5 m, r_1 = 30 km and λ = = 10 cm. For small l below the maximum of the first lobe the change in the intensity of the fluctuations is inversely proportional to h_2^2; for large l the height dependence virtually degenerates. However, in each case there are "noise lobe" minima corresponding to the maxima of the lobes of the average field and vice versa.

* According to our measurements, at elevations of not more than 40 meters l may vary from 3 to 30 meters and more.

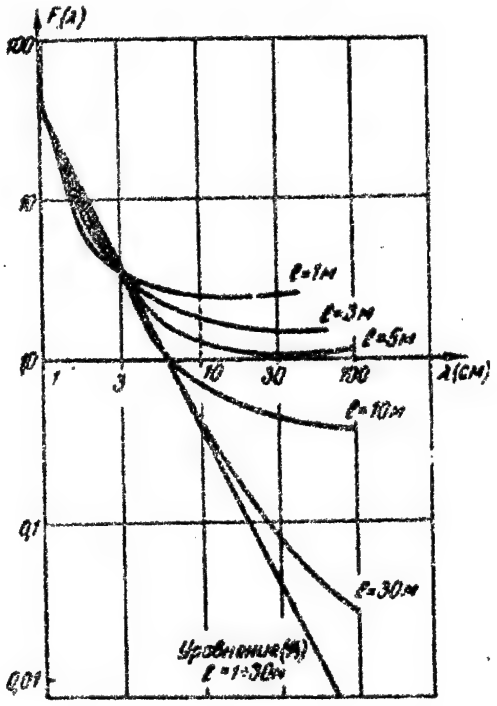

Fig. 3: Frequency characteristic of the intensity of the fluctuations.

$$F(\lambda) = \left| (\overline{\Delta\varphi})^2 10^{12} / \mu^2 l \right| \text{ град}^2 \cdot м^{-1}$$
$$(h_1 = 5\ м,\ r_1 = 30\ км,\ h_2 = 40\ м).$$

In the general case the space delay angle φ_1 depends not only on the wavelength, range and heights h_1 and h_2 but also on the degree of refraction [9]. Consequently, a variation in refraction may also lead to changes in the intensity of the fluctuations. When the refraction increases, the lobes of the average field "sink" (φ_1 increases) permitting a qualitative explanation of the experimentally observed decrease in fluctuation with increase in refraction when working in the region below the maximum of the first lobe. Correspondingly, a change in refraction, leading to an interference minimum of the average field at the receiving station, may cause a sharp increase in the fluctuations without much of a change in the state of the turbulent medium.

3. Thus, the presence of a smooth boundary surface may lead to a substantial increase in the intensity of the fluctuations compared with the case of free space for identical parameters $\overline{\mu^2}$ and of the turbulent medium.

As follows from the formulation of the problem, this effect may be substantially weakened if the mirror character of the reflection ($|R| \ll 1$) is disturbed or if we use high-directional antennas, capable

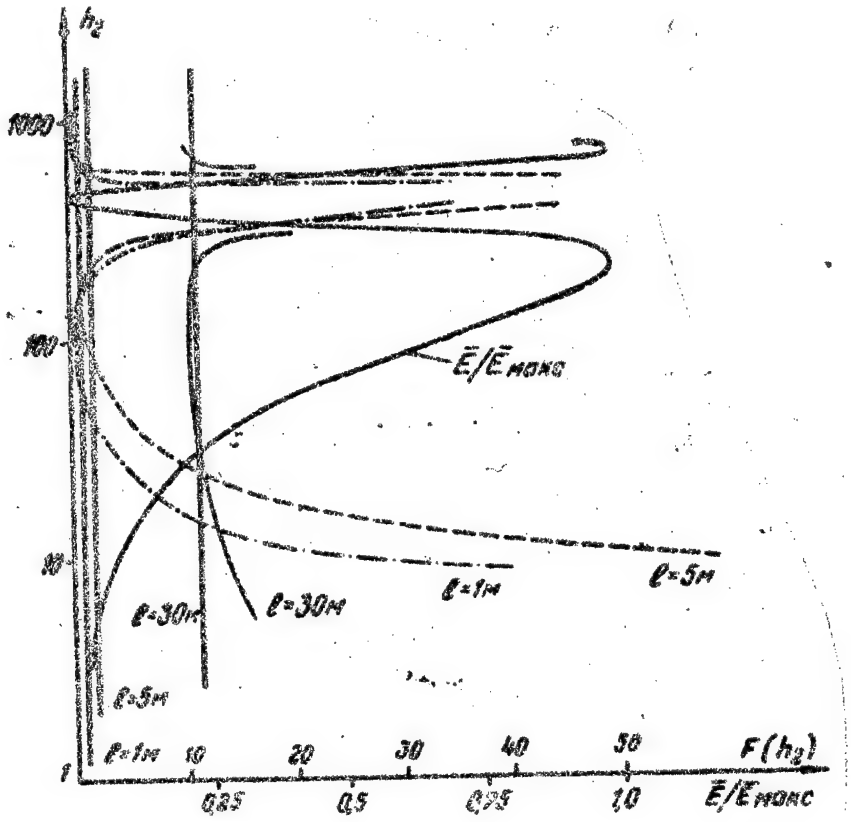

Fig. 4: Height dependence of the fluctuations :

$$F(h_2) = \left[\overline{(\Delta\varphi)^2} 10^{12}/\overline{\mu^2}\right] \text{ degrees}^2$$

(h_1 = 5 m; r_1 = 30 km; λ = 10 cm; straight lines - equation (14).)

of separating the direct and reflected waves. In particular, an increase in the roughness of the boundary surface , leading to a reduction in the intensity of the reflected wave ($|R| \ll 1$), ought to be accompanied by a reduction in the intensity of the fluctuations. This effect may be noted in measuring phase fluctuations over the sea. Ordinarily the greatest fluctuations occur, as a rule, in calm weather when the sea has a mirror-like surface. An increase in the state of agitation of the surface is invariably accompanied by a reduction in the intensity of the fluctuations.

In conclusion, note that the good qualitative agreement between calculation and experiment confirms the necessity of taking into account the effect of a boundary surface on fluctuations. Naturally, the results obtained are imperfect compared with an accurate solution of the problem of the fluctuations in a field in the presence of a boundary surface, making possible an evaluation of the limits of applicability of the approximate method and the accuracy of the numerical estimates made. It would be most desirable to solve the wave equation for a turbulent medium taking into account the boundary conditions for a spherical earth, especially in the region of umbra and penumbra, where the geometrical methods used are inapplicable. Values for phase fluctuations in this region (beyond the limits of the radio horizon), obtained experimentally, reveal a considerably more rapid increase in the fluctuations with distance than in the zone of direct visibility (when $1 \lesssim r/r_{horizon} \ll 2$, $\overline{(\Delta\varphi)^2} \sim r_1^6$).

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ON THE CORRELATION OF FLUCTUATIONS IN SHF RADIO WAVES PROPAGATED IN AN INHOMOGENEOUS MEDIUM.

Pages 395-399

by A.V. Men'

The longitudinal correlation of fluctuations in the near zone is determined; expressions are introduced for the correlation coefficients of fluctuations in phase and amplitude at spaced points for arbitrary orientation relative to the source.

The propagation of waves in an inhomogeneous turbulent medium $n = n(\underline{r},t)$ is accompanied by fluctuations in amplitude and phase, the statistical properties of which are characterized both by intensity (mean squares) and their correlation at differents points in space. Research [1-7] has mainly been directed to determining the intensity of the fluctuations and their correlation along selected directions relative to the communication line - transversely and in the direction of propagation; in the latter case expressions have been determined only for the far zone. It therefore appears worthwhile to publish some results obtained for the case of arbitrary orientation of the spaced points A and B relative to the source of radiation (see Fig. 1a).

1. First we shall determine the correlation of fluctuations in the near zone for propagation along parallel routes of different length.

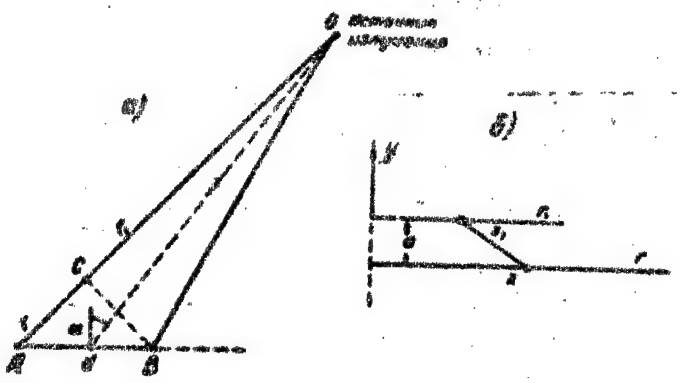

Fig. 1

If the inhomogeneity of the medium is not great, when

$$n = 1 + \mu \qquad (\mu \ll 1), \tag{1}$$

on condition that the change in phase and amplitude for wavelength distances is small, in the near zone the fluctuations in phase ψ_r and amplitude $A(\underline{r})/A_0$ of a plane wave will be respectively [1,2]*:

$$\psi_r = \psi(r) - kr = \frac{2\pi}{\lambda}\int_0^r \mu\, dx;$$
$$\chi_r = \ln\frac{A(r)}{A_0} = -\frac{1}{2}\int_0^r (r-x)\,\Delta_1\mu\, dx, \tag{2}$$

where $\psi(\underline{r})$, $A(\underline{r})$ are the phase and amplitude of the wave, A_0, kr are values of the amplitude and the phase delay in propagation in a homogeneous medium, λ is the wavelength and $\Delta_1\mu$ is the transverse Laplacian operator:

$$\Delta_1\mu = \frac{\partial^2\mu}{\partial y^2} + \frac{\partial^2\mu}{\partial z^2}. \tag{3}$$

On the basis of expressions (2), (3) the correlation functions of fluctuations along two parallel routes of different length (see Fig.

* Direction of propagation taken as the x - axis.

1b) are:

$$\overline{\psi_{r_1}\psi_r} = \frac{4\pi^2}{\lambda^2}\int_0^{r_1} dx_1 \int_0^{r} \overline{\mu(x_1)\mu(x)}\,dx;$$

$$\overline{\gamma_{r_1}\gamma_r} = \frac{1}{4}\int_0^{r_1}(r_1 - x_1)\,dx_1 \int_0^{r}(r-x)\left[\overline{\frac{\partial^2\mu(x_1)}{\partial y^2} + \frac{\partial^2\mu(x_1)}{\partial z^2}}\right]\times$$

$$\times\left[\overline{\frac{\partial^2\mu(x)}{\partial y^2} + \frac{\partial^2\mu(x)}{\partial z^2}}\right]dx. \qquad (4)$$

In the case of an unbounded, statistically isotropic troposphere described by a correlation function of pulsations in the refreactive index n of the form

$$\overline{\mu_1\mu_2} = \overline{\mu^2}\exp(-\Delta r^2 l^{-2}), \qquad (5)$$

where Δr is the distance between points in space and l is a parameter of the correlation function, it is possible to get:

$$\overline{\psi_{r_1}\psi_r} \approx 4\pi^{5/2}\overline{\mu^2}k^{-2}r_1\exp(-a^2 l^{-2}) \qquad (r_1 \gg l) \qquad (6)$$

(a is the distance between the routes).

Analogously, the correlation of amplitude fluctuations is determined in the same approximation from the expression:

$$\overline{\gamma_{r_1}\gamma_r} \approx \frac{\pi}{2}\sqrt{\pi}\,\overline{\mu^2} l^{-3} r_1^2\,(1,\ r r_1^{-1} - 0{,}5)(1 - 2a^2 l^{-2} + a^4 2^{-1} l^{-4})\times$$

$$\times\exp(-a^2 l^{-2}). \qquad (6a)$$

On the basis of expressions (6), (6a) we determine the coefficients of spatial correlation for longitudinal and parallel spacing: when a = 0

$$R_{\parallel} = \overline{\psi_{r_1}\psi_r}/\sqrt{\overline{\psi_{r_1}^2}}\sqrt{\overline{\psi_r^2}} = \sqrt{r_1/r}\ ; \qquad R_{\perp} = \sqrt{(r_1/r)^3}\,(1{,}5\,r/r_1 - 0{,}5) \qquad (7)$$

and, correspondingly, when $r_1 = r$ *

---

* Analogous expressions for transverse correlation coefficients are presented in [4] and [7].

$$R'_\varphi = \exp(-a^2 l^{-2}); \qquad R'_\gamma = \exp(-a^2 l^{-2})(1 - 2a^2 l^{-2} + a^4 2^{-1} l^{-4}). \quad (8)$$

where R_φ, R_γ are the coefficients of longitudinal correlation of the phase and amplitude fluctuations, R'_φ, R'_γ are the corresponding coefficients of transverse correlation.

The expressions obtained for the correlation coefficients are illustrated in Fig. 2. It should be noted that if the correlation of the amplitude fluctuations decreases more rapidly in transverse spacing, the opposite holds true for longitudinal spacing, when the phase fluctuations become more rapidly decorrelated. For example,

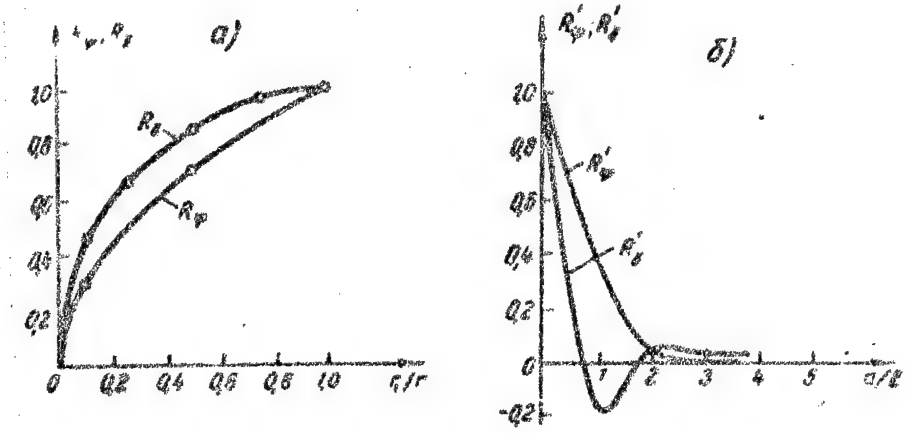

Fig. 2

for longitudinal spacing the correlation coefficient decreases about 0.5 by 0.88 r for amplitude fluctuations and by 0.75 r for phase fluctuations respectively, where [2]

$$r \leqslant 0{,}5\,\pi l/\lambda. \quad (9)$$

2. We shall now pass to the determination of the correlation of fluctuations at points A and B in space arbitrarily oriented relative to the source of radiation (see Fig. 1a). Taking into account the expressions obtained for longitudinal correlation, we assume, as an approximation, complete correlation of the fluctuations at points A and C for $d \ll r$, where d is the distance between the points and

C is the projection of B on the direction OA *.

Then, neglecting the difference in the intensity of the fluctuations in these points, i.e. assuming $\overline{\psi_A^2} \approx \overline{\psi_C^2} = \overline{\psi^2}$ и $\overline{\gamma_A^2} \approx \overline{\gamma_C^2} = \overline{\gamma^2}$, we have:

$$\overline{\psi_A \psi_B} / \sqrt{\overline{\psi_A^2}} \sqrt{\overline{\psi_B^2}} \approx \overline{\psi_C \psi_B} / \overline{\psi^2} = R'_\psi(d\cos\alpha);$$
$$\overline{\gamma_A \gamma_B} / \sqrt{\overline{\gamma_A^2}} \sqrt{\overline{\gamma_B^2}} \approx \overline{\gamma_C \gamma_B} / \overline{\gamma^2} = R'_\gamma(d\cos\alpha), \qquad (10)$$

where R'_ψ (d cosα) and R'_γ (d cosα) are the coefficients of transverse correlation of the phase and amplitude fluctuations for points spaced at a distance d cosα.

Note that expression (10) also holds true for the far zone ($r > 0.5\pi l^2/\lambda$), since in this case, in accordance with [6,7], for longitudinal spacing decorrelation sets in even more slowly than in the near zone.

Using the expressions for coefficients of transverse correlation presented in [4,6], we shall define the correlation of phase fluctuations in both zones and of amplitude fluctuations in the far zone for intersecting routes (see Fig. 1a) and arbitrary orientation of the base relative to the source when $d \ll r$ in the form:

$$R_{\psi_{AB}} = R_\gamma = \frac{\sqrt{\pi}}{2} \frac{\operatorname{erf}(d\cos\alpha/l)}{d\cos\alpha/l}. \qquad (11)$$

It follows from (11) that, by measuring the correlation coefficient of the fluctuations for different orientations of the base relative to the source, we can determine for given ratios d/λ experimental values for the "scale of the inhomogeneities", characterizing an inhomogeneous medium in average terms.

3. In using phase fluctuations for determining the parameter l analogous results can be obtained more simply by measuring fluctuations in the phase difference (instead of determining the correlation

* When $d/r \leq 0.01$, for example, the error in such an assumption is less than 0.5% for phase and 0.02% for amplitude fluctuations.

of phase fluctuations at the spaced points). In this case, by determining the mean square of fluctuations in the phase difference for different orientations of the base, normalized with respect to its maximum value, we have in both zones:

$$\Phi(\alpha, d/l) = \frac{\overline{(\psi_A - \psi_B)^2}}{(\psi_A - \psi_B)^2_{\max}} = \frac{1 - R_{AB}}{1 - R_{AB\,\min}} = \frac{1 - (\sqrt{\pi}\, l/2d \cos\alpha)\operatorname{erf}(d\cos\alpha/l)}{1 - (\sqrt{\pi}\, l/2d)\operatorname{erf}(d/l)} \quad (12),$$

where $R_{AB_{min}}$ is the minimum value of the correlation coefficient corresponding to $\alpha = 0$. Graphs of this function are given in Fig. 3 for different values of the ratio d/l.

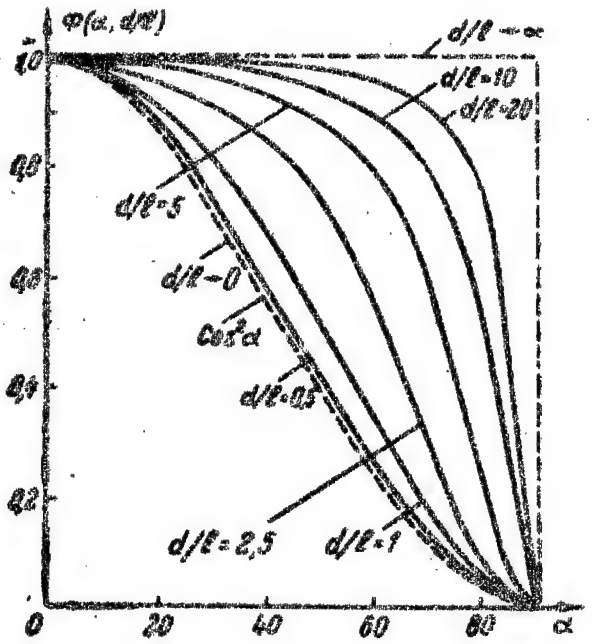

Fig. 3

Measuring the parameter l by rotating a base of fixed length is also possible in this case for a given interval of ratios d/l, for example, when

$$0.5 < d/l < 20.$$

Actually, when $d/l \ll 1$ with an error of about $0.3\, d^2 l^{-2} \sin^2\alpha$

$$\Phi(\alpha, d/l) \to \cos^2\alpha, \quad (13)$$

i.e. in this case information about the parameter l is excluded. In the other extreme case, when $d/l \gg 10$, for $0 < \alpha < \alpha_{rp}$

$$0 < \alpha < \alpha_{rp}$$
$$\Phi(x, d/l) \approx 1 - \sqrt{\pi} l/2d\cos\alpha. \qquad (13a)$$

An important change in the intensity of the fluctuations, greater than 10%, for example, will then be observed in the narrow sector of angles:

$$\alpha_{rp} < \alpha < 90°, \qquad (14)$$

where $\alpha_{rp} = \arccos(5\sqrt{\pi} l/d)$,

which leads to an increase in the errors of measurement.

In conclusion we note that the expressions presented are valid for the probability form of the correlation function of μ (5); however, analogous results may be obtained for different forms of this function, if the correlation of the pulsations at different points in the medium decreases monotonically with distance.

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AN ANALOG STUDY OF THE EFFECTS OF NOISE ON A SELF-EXCITED OSCILLATOR WITH STRONG FEEDBACK

Pages 400-407

By P.S. Landa

An electronic analog has been used to find asfunctions of the noise level the mean times taken for self-excited oscillations to build up and to die away in an oscillator having strong feedback. The results are found to agree well with those predicted from theory.

This paper continues the work presented previously [1] on certain aspects of the effects of regular and random signals on a strongly excited oscillator. We [1] have derived approximate expressions for the probabilities of transition to and from the excited state and for the mean dwell times for the two states; the expressions apply if certain restrictions are imposed on the external signals and on the parameters of the system. It is of interest to see to what extent those [1] results are usable in practice and to obtain new experimental information for values of the parameters etc. not complying with the restrictions (for cases to which the theory may not apply). Electrical analogs can give us that information, and I have examined the problem outlined above by means of an electronic analog machine. To this end I have used as the theoretical expression the equation for tuned-grid oscillator free from grid current and plate-load reaction, my object being to be able to compare the results from the analog

with those from theory. The grid characteristic has been approximated by means of a fifth-power polynomial.

Previously I have pointed out [2] the advantages of using an analog rather than an actual oscillatory system for studies of this kind. The objects of my work have been to verify that electrical analogs may be used in solving problems of statistics and to develop methods of making statistical measurements at very low frequencies as well as to obtain results for a particular oscillator.

1. Apparatus

I used a normal d.c. integrator containing nonlinear units in conjunction with sources of low-frequency noise and sine waves. The noise source was one made by Romañovskiy [3] at the Physics Faculty of Moscow State University; the sine-wave source was an NG-2 oscillator.

The self- excited system represented by the analog was one whose differential equation is

$$\ddot{z} + \omega_0^2 z = 2\delta(z)\dot{z} - \omega_0^2 E \sin \omega_0 t - \omega_0^2 \zeta(t), \qquad (1)$$

in which $\delta(z) = -\lambda(1 - 4\alpha z^2 + 8\beta z^4)$, and $\zeta(t)$ is the noise at the input to the system.

The machine coordinates and time [6] are

$$V_1 = z; \quad V_2 = \dot{z}/\omega_0; \quad V_E = -E; \quad V_\zeta = -\zeta(t); \quad \tau = t$$

(in which case the V all have the same dimensions and the limit cycle of the system takes the form of a circle).

We may put (1) in the form of two equations suitable for use with the analog, namely

$$\frac{dV_1}{d\tau} = \omega_0 V_2;$$

$$\frac{dV_2}{dt} = -\omega_0 V_1 - 2\delta V_2 + 8\delta\,(\alpha V_1^2 - 2\beta V_1^4)\,V_2 + \omega_0 V_E \sin\omega_0 t + \omega_0 V_\xi' . \qquad (2)$$

Figure 1 shows the block diagram of the analog.

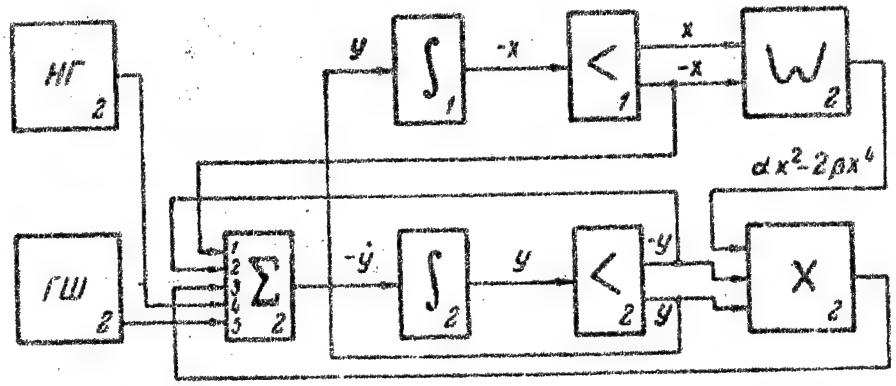

Fig. 1. Block diagram of the analog to a strongly excited oscillator.

The parameters to be chosen for the analog in accordance with (2) are a) the time-constants $RC_I = 1/\omega_0$ of the integrators; RC_{II} can be assigned any value (in my case it was 0.1507 sec); and b) the gains with respect to the inputs to appear in the summator, namely $k_1 = \omega_0 RC_{II}$; $k_2 = 2\delta_0 RC_{II}$; $k_4 = 8\delta_0 RC_{II} K$;

$$k_4 = \omega_0 V_E RC_{II}/V_E'; \quad k_5 = \omega_0 V_\xi' RC_{II}/V_\xi' .$$

in which K is the product of the attenuation factors of the multiplying (nonlinear) units, V_E' is the amplitude of the signal from the sine-wave source, and V_ξ' is the signal (voltage) from the noise source. The factors k_2 to k_5 can be adjusted in accordance with any change in the parameters or noise level.

2. Noise-Level Measurements

Quantitative results that can be compared with the theory are obtained only if we measure the noise level λ^{-1}, which is related to the spectral noise density $2x(\omega_0)$ at a frequency ω_0 by [1] the relation

$$\lambda^{-1} = \omega_0^2 x(\omega_0)/4. \qquad (3)$$

I measured the noise level by means of the system shown as its block diagram in Fig. 2. The signal from the noise source is fed to a narrow-band linear resonant circuit whose

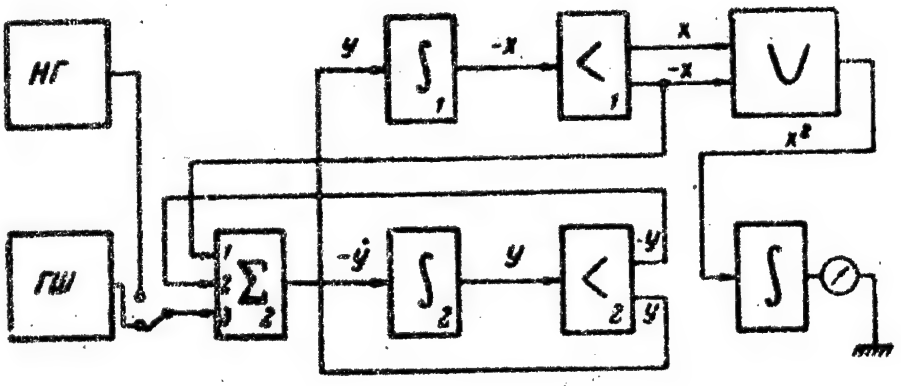

Fig. 2. Block diagram of the analog used in the noise-level measurements

frequency response is $F(\omega)$. The voltage appearing at the output of that circuit is squared by one of the nonlinear units, the square being integrated for a time T_1. The integrator's output voltage is

$$U_1 = 2\gamma T_1 x(\omega_0) RC_{11}^2 \omega_0^2 k_5^{-2} \Delta f, \qquad (4)$$

in which Δf is the equivalent passband of the circuit, which is defined by

$$\int_0^\infty F^2(\omega)\,df = F^2(\omega_0)\,\Delta f.$$

in which γ is the transfer factor at frequency ω_0 for the system as a whole. That factor may be measured on the system as it stands by supplying a sine-wave signal whose frequency is ω_0 and whose amplitude E_0 is known. Then the integrator's output voltage will be

$$U = \gamma T E_0^2/2, \qquad (5)$$

in which T is the integration time. We combine (4) and (5) to get that

$$\varkappa(\omega_0) = U_1 T E_0^2 k_s^2/4UT_1 RC_{11}^2 \omega_0^2 \Delta f. \qquad (6)$$

We see that (3) and (6) give us that the noise level is

$$\lambda^{-1} = k_s^2 U_1 E_0^2 T/16\, UT_1 RC_{11}^2 \Delta f. \qquad (7)$$

All quantities appearing in this formula are accessible to direct measurement.

I made measurements at various noise levels, for which purpose the input to the summator was kept constant, any change being made in terms of the corresponding gain. This approach had the advantage that the errors arising from measurements of the noise level (which can be as large as 25%) are excluded, since the gain can be measured with an error of only 1%.

3. Mean Dwell Time in the Excited or Quiescent State

There are two ways of measuring the mean times spent in the excited and quiescent states. The first is to measure either such time in terms of the mean of a large number of separate measurements of the time the initially excited (quiescent)

oscillatro remains in the excited (quiescent) state. Those measurements give us the probability of a transition from the excited state to the other state during a time $\tau \leqslant t$ as

$$p(t) = m(t)/N, \qquad (8)$$

in which m(t) is the number of measurements in which the time was found to be $\tau \leqslant t$ and N is the total number of measurements. The probability q(t) of the reverse transition during a time $\tau \leqslant t$ is expressed by a similar formula. Now theory [1] shows that

$$p(t) = 1 - e^{-t/t_{2cp}}; \quad q(t) = 1 - e^{-t/t_{1cp}}, \qquad (9)$$

in which t_{1m} is the mean duration of a train of oscillations and t_{2m} is the mean duration of a quiescent period.

Provided that N is adequately large, we can verify by experiment that (8) and (9), which are exponential in form, represent (perhaps within certain limits only) the actual probability distributions.

The mean deviations of the measured mean times and of p(t) and q(t) are controlled by N, being respectively

$$\sqrt{\frac{1}{t_{cp}^2}\left(\frac{t_1+t_2+\cdots+t_N}{N} - t_{cp}\right)^2} = \sqrt{\frac{1}{N}}; \quad \sqrt{\frac{1}{p^2}\left(\frac{m}{N} - p\right)^2} = \sqrt{\frac{1-p}{Np}}.$$

I recorded the duration of each of the two states fifty times for each noise level; the error arising from the finite number of readings is about 15%, the error (in %) in p itself being $15\sqrt{(1-p)/p}$.

Figure 3 shows t_{1m} and t_{2m} as function of λ^{-1} for E = 0, $\omega_0^2 = 98$ sec^{-2}, $\delta = 0.0895$ sec^{-1}, $\alpha = 2.307 \times 10^{-3}$ v^{-2}, and $\beta = 0.985 \times 10^{-6}$ v^{-4}. The amplitudes of the unstable and stable limit cycles were $R_1 = 24$ v and $R_2 = 42$ v respectively.

To establish the extent to which the system corresponded to the specified values of the parameters, I recorded an oscillogram of the characteristic $f(V_1) = \alpha V_1^2 - 2\beta V_1^4$, which I compared with the theoretical curve. The two curves differed at no point by more than 10%. The deviations of the real parameters from the specified ones cause a certain systematic error in the mean times, which error does not depend on the noise level.

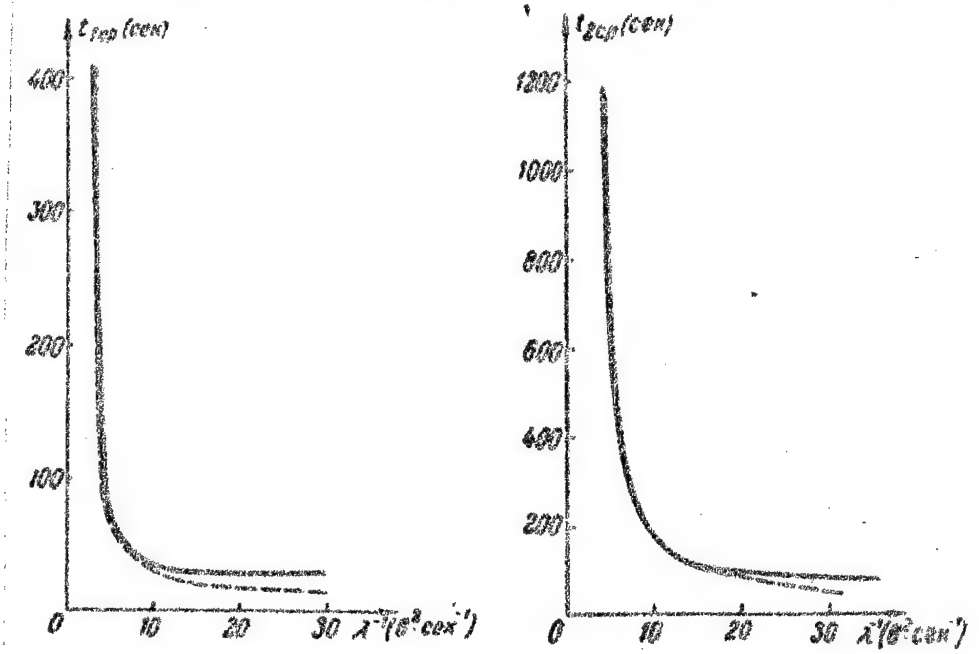

Fig. 3. Relation of dwell time in a state to noise level a) Quiescent state, b) excited state; the full lines derive from theory, the broken ones from the measurements.

The results obtained in [1] are correct if $\lambda^{-1} \ll \delta/\alpha$, or if $\lambda^{-1} \ll 39\ v^2\ \text{sec}^{-1}$ in the case of the parameters quoted above. Figure 3 and Table 1 show that the mean deviation of the theoretical values from the measured ones is about 10% when $\lambda^{-1} < 10\ v^2\ \text{sec}^{-1}$, although that deviation becomes much larger (about 30 - 40%) at higher noise levels.

Table 1

A^{-1} (v^{-2} sec^{-2})	t_{1m} (sec)		t_{2m} (sec)		$\Delta t_{1m}/t_{1m}$ (%)	$\Delta t_{2m}/t_{2m}$ (%)
	Theory	Expt.	Theory	Expt.		
2.81	341	401	4250	—	+ 17	—
4.23	105	102	958	1070	— 2.8	+ 11.7
6.00	55.8	55	405	417	— 1.4	+ 3
8.48	38.1	34	222.3	237	— 10.7	+ 6.6
12.70	20.6	22	137.6	128	— 26	— 7
16.90	27.3	19	108.3	94	— 30	— 13.2
25.40	20.7	15	84.5	63	— 44	— 25.5

Figures 4 and 5 show p (t) and q(t) for low and high noise levels; the broken lines represent the p(t) and q(t) predicted from (9). We see that q(t) deviates from the exponential distribution law by amounts exceeding the limits of error when the noise level is high, whereas p(t) continues to obey that law. The cause is that the exponential law applies provided that the time spent by the image point in the low-probability region (near the threshold of excitation) is much less than the mean time spent in the quiescent state S_1 or in the excited state S_2 (see [1]). Now $t_{1m} \ll t_{2m}$, so that condition is violated in S_1 much sooner than in S_2.

The second method of measuring the mean times (which gives us only the ratio of t_{1m} to t_{2m}, however) employs expressions derivable from ones given in [1], namely

$$t_{1cp}/(t_{1cp} + t_{2cp}) = \int_{S_1} A\, W(A, \varphi)\, dA\, d\varphi ; \tag{10}$$

$$t_{2cp}/(t_{1cp} + t_{2cp}) = \int_{S_1} A\, W(A, \varphi)\, dA\, d\varphi , \tag{11}$$

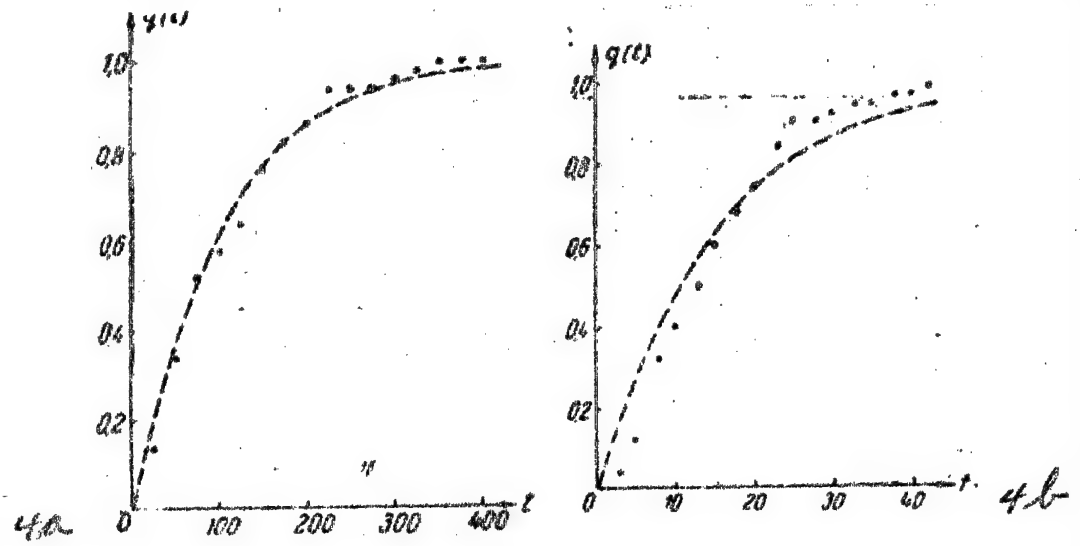

Fig. 4. Excitation probability.
a) For $\lambda^{-1} = 4.23\ v^2 sec^{-1}$, b) for $\lambda^{-1} = 25.4\ v^2 sec^{-1}$;
the broken lines denote the theoretical curves $1 - exp\ (-t/t_m)$

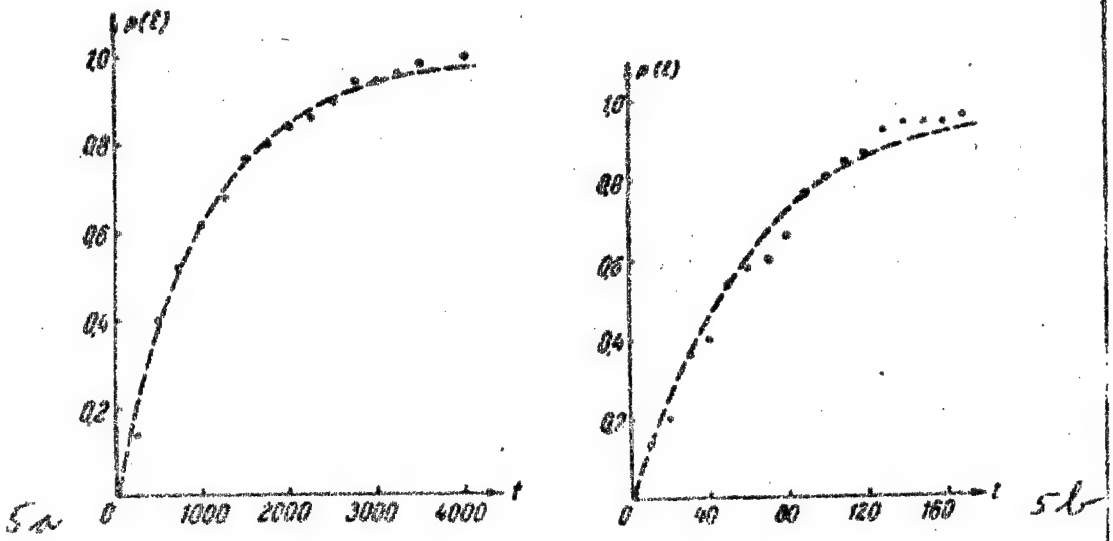

Fig. 5. Quenching probability.
a) For $\lambda^{-1} = 4.23\ v^2 sec^{-1}$, b) for $\lambda^{-1} = 25.4\ v^2 sec^{-1}$;
the broken lines denote the theoretical curves $1 - exp(-t/t_m)$

in which $W(A, \varphi)$ is the stationary probability distribution satisfying the null boundary condition.

Measurements of that distribution (see [4, 5]) can be made by means of photometric methods applied to a photographic recording made over and adequately long time (a time much longer than $t_{1m} + t_{2m}$) of the phase representation of the system (Fig. 6) In fact it was difficult to make a recording extending over a period so long, because quite exceptional measures would have been needed in order to balance out the drift of the zero. Therefore I recorded several pictures whose total exposure time T was 11 hr ($t_{1m} + t_{2m}$ being 472 sec). The photometer results obtained on each pattern were averaged; the probability distribution so obtained satisfies the condition that it must vanish at infinity (not the null boundary condition). The deviation from the proper distribution is slight in the high-probability range if the noise level is low. As a result, the error in the mean times is quite small.

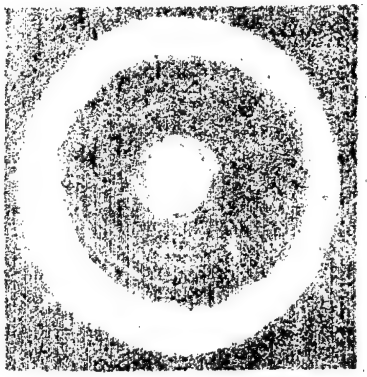

Fig. 6. Phase pattern recorded with a one-hour exposure.

I made measurements with $\lambda^{-1} = 6\ v^2 sec^{-1}$, so the condition $\lambda^{-1} \ll \delta/\alpha$ was not obeyed very well ($\delta/\alpha \approx 39\ v^2 sec^{-1}$). The result is that the t_{1m}/t_{2m} obtained in the two ways do not agree at all well (the first way gave $t_{1m}/t_{2m} = 55/417 = 0.132$, whereas the second gave 0.256). The t_{1m}/t_{2m} given by the theoretical distribution satisfying the condition that it must be zero at infinity is 0.212, which differs by about 20% from both of the measured values.

I also measured t_{1m} and t_{2m} when an external sine-wave signal was present, but the mistuning could not be kept exactly zero on account of frequency instability in the source, so the agreement with theory was only qualitative. For example, the values I obtained for $E = 0.126$ v and $\lambda^{-1} = 6\ v^2 sec^{-1}$ where $t_{1m} = 42$ sec, $t_{2m} = 446$ sec, whereas those found in the absence of the signal were $t_{1m} = 56$ sec and $t_{2m} = 405$ sec. These results do show, however, that the signal facilitates excitation.

My results indicate that the theoretical methods of calculating build-up and decay times for strong-feedback oscillators are applicable even though the inequalities imposing restrictions on the inputs are not complied with very rigorously. Theory gives an over-estimate of the times if the noise level is high.

In conclusion I wish to thank S.P. Strelkov for much assistance in this work and diploma student V.B. Skomorokhov for performing part of the experiment.

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EXPERIMENTAL INVESTIGATION OF THE EFFECT OF RANDOM INTERFERENCE ON SELF-OSCILLATOR SYNCHRONIZATION PROCESSES.

Pages 408-419

by I.G. Akopyan

Phase and amplitude fluctuations are investigated for a synchronized self-oscillator under the influence of random interference. Methods of directly measuring the fluctuations in phase and amplitude and drift from the mean frequency are worked out. The experiment is based on the application and development of a method of studying random processes with the aid of a cathode-ray tube. In addition to steady-state conditions the transient process following the cutting in of the synchronizing signal is studied. The experimental results are compared with theoretical calculations made by a number of authors.

Introduction

The study of the influence of fluctuation noise on a synchronization process is of practical interest in connection with the use of synchronization in frequency stabilization, coherence technique and other devices. Different aspects of the problem have been studied theoretically in references [1-5]. In [1] Rytov, using the small parameter method and correlation theory, examined the behavior of a self-oscillating system under the action of an harmonic locking force and in the presence of small fluctuations in the system. Rytov finds values for the steady-state dispersion of phase and amplitude fluctuations in a self-oscillator and the energy ratio the discrete line and continuous portion of the self-oscillator spectrum. In [2] Rayevskiy and Khokhlov consider the phase fluctuations of a self-oscillator, acted on by a small harmonic force and quasi-monochromatic interference. The authors found the conditions for maintaining the synchronized regime and evaluated the phase dispersion for small amounts of detuning. The article by Stratanovich [3] differs from

the preceding papers in that the Einstein-Fokker method is employed, enabling the author to make an investigation without being limited to the case of small phase fluctuations and to find the laws of distribution of the phase and amplitude fluctuations and also the displacement of the mean frequency of the self-oscillations due to the fluctuational effects. In articles [4,5], to which the same limitations as above apply, the linearization method was employed to study the transient processes in a self-oscillator, subject to the action of fluctuation noise, following the cutting-in of the synchronizing harmonic signal; the behavior of transient dispersion of phase and amplitude fluctuations was investigated and a study was made of the effect of a random initial phase. Note that the problem of establishing the phase of a self-oscillator, synchronized with an harmonic signal with random initial phase (without fluctuational effects), has already been studied by Kobzarev* [6] under similar conditions and, for the more general case, by Lisitsian [7] with numerical integration of the shortened equations.

The aim of the present work is the experimental study of the influence of random interference on continuous and pulsed synchronization processes.

1. Experimental setup

The object of investigation was a low-frequency tube oscillator of the Thompson type with a grid circuit, working at a frequency of 40 kc. The (continuous or pulsed synchronizing signal and the fluctuation noise were fed directly into the oscillator circuit. We therefore introduced into the capacitive and inductive branches of the circuit small resistances, to which we fed the signal and the noise from special multi-tube cathode followers with output resistances of less than 10 ohms. A noise diode with a three-stage low-frequency amplifier was used as the noise generator. The frequency characteristic of the amplifier was chosen so that the maximum amplification corresponded to the frequency of the oscillator being investigated (40 kc), while the bandwidth was about 50 kc. This made it possible, under the conditions of the problem, to consider the active noise as being "white".

The experimental setup described below enabled us to measure the unidimensional laws of distribution of the phase and amplitude of the self-oscillations as well as their mean frequency and the difference between the latter and the frequency of the forcing signal. We shall now consider the measuring method in greater detail.

a) Method of measuring fluctuations in phase.

The block diagram for phase measurement is set out in Fig. 1.

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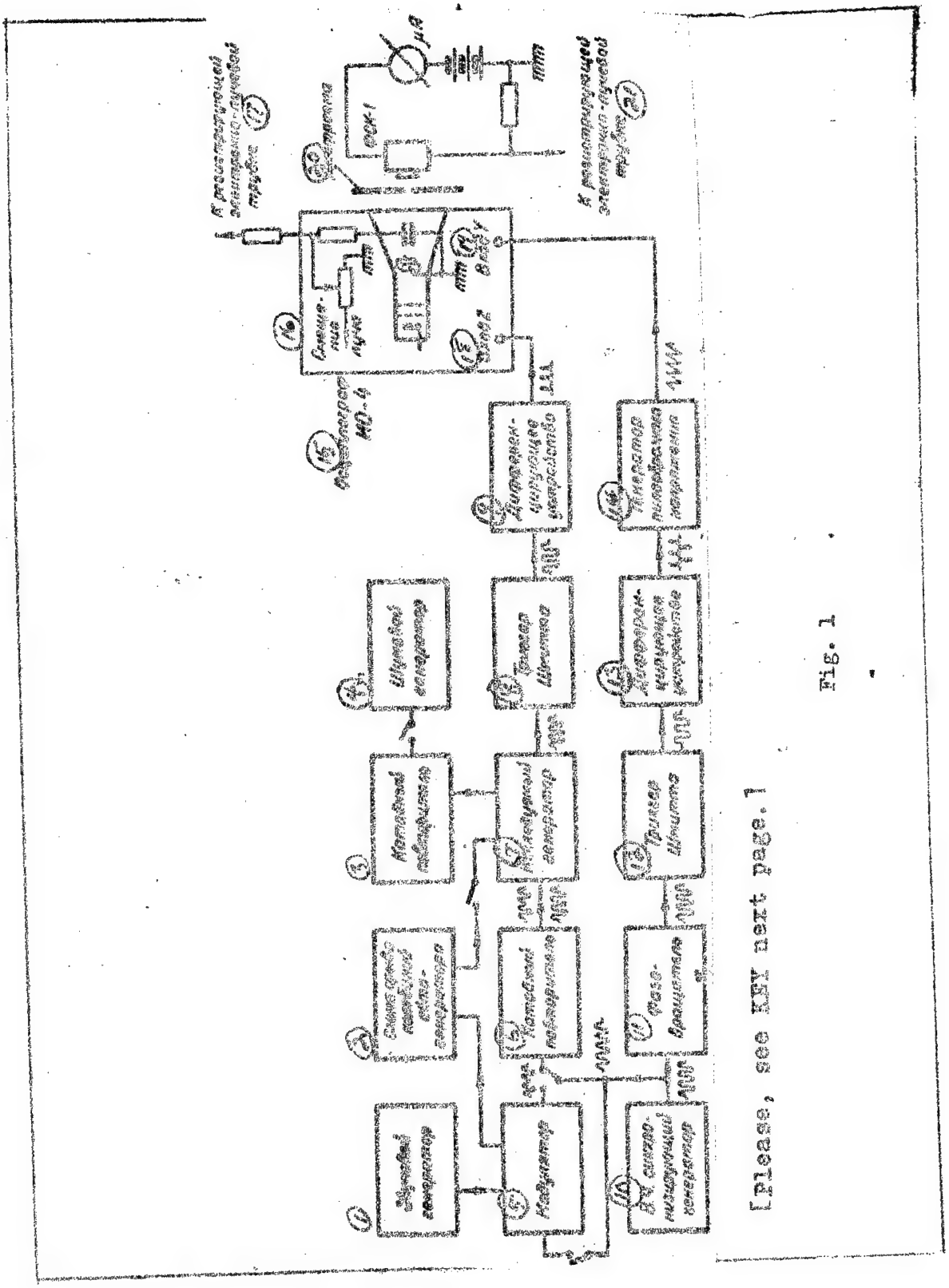
* Unfortunately, this was published in an obscure journal and only came to our attention after a considerable delay.



[Please, see KEY next page.]

Fig. 1

KEY to Figure 1:

1) A-F oscillator;
2) Oscillation stopping circuit;
3) Cathode follower;
4) Noise generator;
5) Modulator
6) Cathode follower
7) Test oscillator
8) Schmitt trigger
9) Differentiating device
10) H-F synchrooscillator
11) Phase-shifter
12) Schmitt trigger
13) Differentiating device
14) Saw toothed voltage generator
15) LO-4 oscillograph
16) Beam displacement
17) To recording cathode-ray tube
18) Input Z
19) Input Y
20) Diaphragm
21) To recording cathode-ray tube

A synchronizing signal, taken from a 101-I generator, acts on the test oscillator across a cathode follower. At the same time the synchronizing signal is fed through a phase-shifter to a Schmitt trigger, which tips over twice per period, operating at the moment of zero-transition of the control voltage. Thus, the trigger generates a square wave with the frequency of the forcing signal, rigidly tied in phase to the synchronizing voltage. This square wave is then differentiated, the negative horns being limited, while the narrow positive horns (duration 0.3 microsecs.) are fed as synchronizing pulses to a blocking oscillator. The blocking oscillator pulses act on a discharge tube, forming a saw-tooth wave with good linearity and very small reversal (less than 0.2 microsecs.). Thus, a saw-toothed voltage, having the frequency of the forcing signal and tied to it in phase, is generated, while the phase-shifter varies the phase displacement within the limits 0 to 2π. The saw-toothed voltage thus obtained is fed to the Y input of an I0-4 oscillograph.

The voltage from the self-oscillator circuit across the by-pass cathode follower is passed to another Schmitt trigger, which also functions in the way described above. The narrow pulses obtained by differentiating the square wave are led to the Z input of the I0-4 oscillograph, producing bright spots on the saw-toothed voltage. When the noise is not acting and we have synchronization, the bright spots all fall on the same phase of the saw-toothed voltage and the entire series of spots lies on a straight horizontal line, the position of which is determined by the constant phase shift φ_0 between the forcing signal and the oscillations of the self-oscillator (oscillogram, Fig. 2a). When the fluctuations act at the same time as the synchronizing signal, the phase displacement varies in accordance with a random law, fluctuating about the steady-state value φ_0. In certain cases irreversible phase jumps of a period occur (see [3,7]), leading to a displacement of the mean frequency of the oscillations of the self-oscillator relative to the forcing signal. An oscillogram of phase fluctuations about the steady-state value is shown in Fig. 2b .

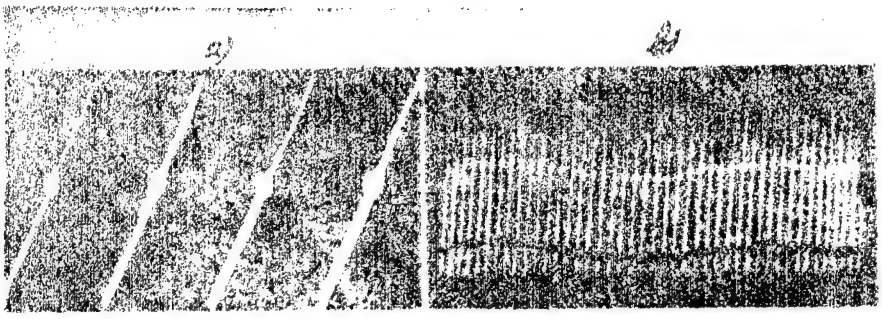


Fig. 2

For the quantitative evaluation of the phase fluctuations we took the unidimensional distribution law $w(\varphi)$. The measurements are based on the very fruitful cathode-ray tube method [8-10]. Let us briefly consider the measuring technique. In front of the oscillograph screen we set up a photoresistance with a diaphragm, enabling us to record, in terms of the magnitude of the current flowing through it, values of the average brightness of points on the screen at different levels of the saw-toothed voltage (for different φ), proportional to the probability density $w(\varphi)$ sought. As taking readings with respect to points is a long and laborious process, an automatic record of the distribution law is obtained by using an additional cathode-ray tube [9], the distribution law $w(\varphi)$ being photographed directly from its screen to a known scale. Fig. 3 shows how this device works. On displacing the electron beam in the main oscillograph with respect to the vertical there is a synchronous displacement of the beam, across the diaphragm and the photoresistance, in the auxiliary cathode-ray tube, the plates of which are connected at constant voltage with those of the main tube, while a voltage proportional to the current passing through the photoresistance is supplied to the second pair of plates. As a result, the beam of the second tube traces the required distribution law $w(\varphi)$ on the screen, where it is photographed.

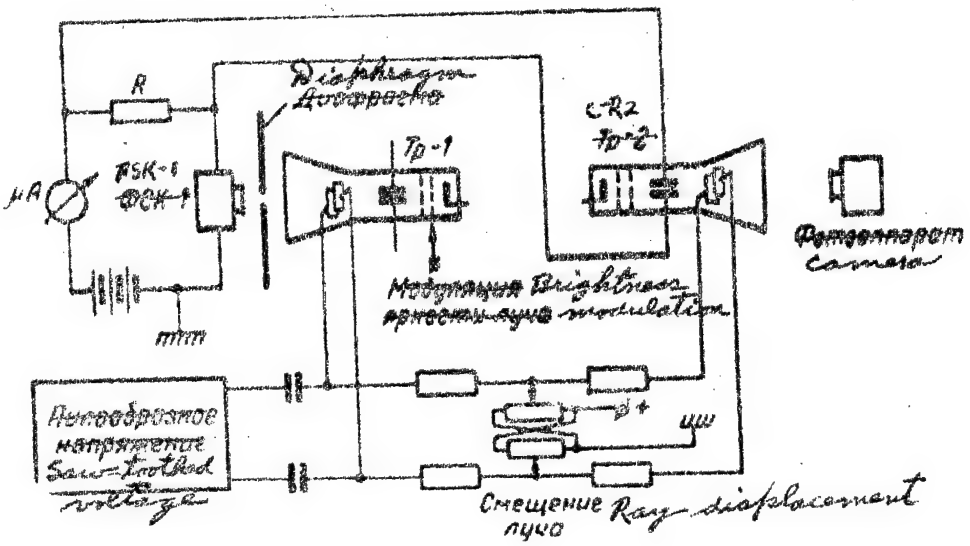


Fig. 3

b) Method of measuring amplitude fluctuations.

The block diagram for amplitude measurements differs from that shown in Fig. 1 in that the oscillations of the oscillator being tested are fed across an output cathode follower directly to the Y-input of the oscillograph and at the same time across a four-period delay line (τ_{delay} = 6.25 microsecs.) to a Schmitt trigger, the pulses from which are differentiated and fed to the Z-input of the oscillograph. As a result the peaks of the voltage being investigated are brightened. In the absence of noise effects all the bright spots lie on a single horizontal line (Fig. 4a), the height of which corresponds to the amplitude of the self-oscillations. Introducing fluctuational effects led to an erosion of the deltoid amplitude distribution to a certain finite distribution $w(A)$, which was also measured by the cathode-ray tube method. Fig. 4b shows an oscillogram for a self-oscillator subjected to fluctuational effects.

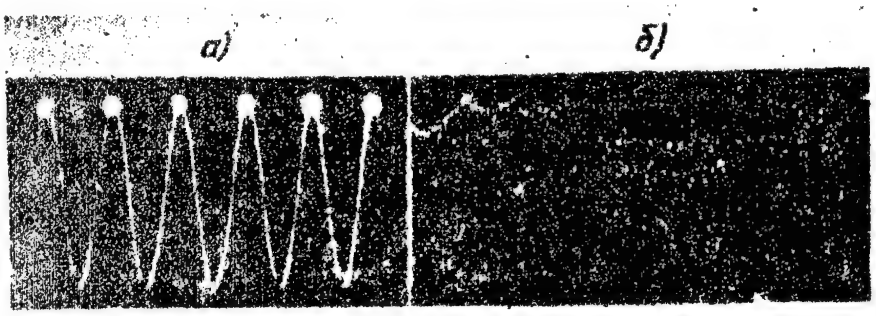


Fig. 4

c) Investigation of transient processes.

The processes of establishing phase and amplitude in a synchronized self-oscillator were investigated by means of the same setup, with the sole difference that the test oscillator was synchronized not by means of a continuous radio signal but but by means of square pulses, formed from continuous oscillations of the 101-I generator in a special balanced modulator with a very short settling time compared with the period of the oscillations. A slave sweep voltage, triggered by the leading edge of the modulator pulse, was supplied to the horizontal deflection plates of the main oscillograph (see Fig. 3). The duration of the sweep was calibrated with the aid of time markers. The distribution laws were recorded exactly as before but at different sections of the sweep, corresponding to different moments of time after cutting in the synchronizing signal.

Considerable difficulty was experienced in establishing experimentally random initial phase of the synchronizing pulses relative to the self-oscillations. It can be shown that even when the repetition

rate of the radio pulses is not a multiple of the carrier frequency, in the absence of noise effects a constant initial phase difference, not varying from pulse to pulse, is established and the presence of fluctuations leads only to a spread about this phase value. In order to get uniform initial phase distribution, it was necessary to use a special oscillation stopping circuit at the end of each synchronizing pulse, as a result of which (when the repetition rate was not a multiple of the carrier frequency) the initial phase varied linearly from pulse to pulse, going through all the values between 0 and 2π. To illustrate this Fig. 5a shows an oscillogram for the establishment of the phase of an oscillator, synchronized by means of radio pulses with random initial phase, in the absence of noise. Fig. 5b shows the same process in the presence of noise. By taking photometric readings of such a pattern, averaged over sufficiently long intervals, at different sections t = const., we obtained the transient phase distribution law $w(\varphi,t)$. In just the same way we measured the transient amplitude distribution law $w(A,t)$.

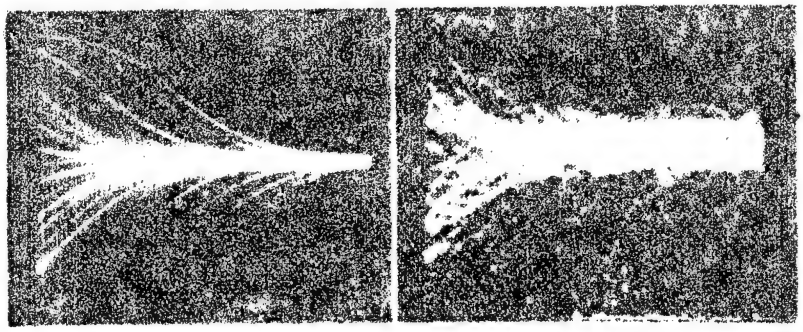


Fig. 5

d) Method of measuring frequency.

The irreversible phase jumps of a period, due to fluctuations, lead to a displacement of the mean frequency of the self-oscillator relative to the frequency of the forcing signal. The relative magnitude of this displacement is rather small, since it constitutes only a fraction of the synchronization band. Hence it is clear that the frequency must be measured with sufficiently high accuracy, in order that the measuring error does not obscure the effect being investigated. Thus, when the synchronizing bandwidth is of the order of 10^{-2} of the carrier and the frequency displacement effect of the order of 10^{-1} of the synchronizing band, it is necessary to have an accuracy of frequency measurement of not less than 10^{-4}. Moreover, as a result of the random nature of the phase jumps, the frequency displacement has a statistical character and cannot be detected by separating regular

beats between the two frequencies. All this determined the method of frequency measurement, the block diagram for which is shown in Fig. 6. The oscillations of the test oscillator are converted into pulses

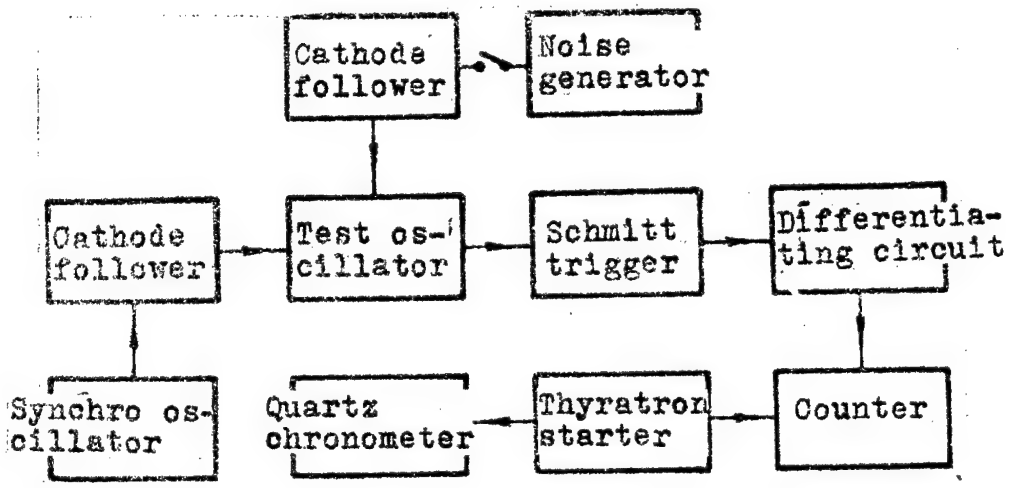


Fig. 6

which are fed to the input of an electronic counter. The electronic counter is started and stopped in time with a quartz chronometer by means of starting and stopping pulses, generated by a special thyratron starter, actuated by depressing a push-button. The quartz chronometer consists of an electronic counter, which counts the number of periods of a high-stability quartz oscillator with a period of 0.25 microsecs. The frequency measurements are made by cutting in the self-oscillator period counter and the quartx chronometer for 20 - 30 seconds, after which the mean frequency of the self-oscillations is calculated from the readings with great accuracy (10^{-5}). The period counter was a semiconductor triode binary counter with a resolving power of 10^{-5} secs. and a single-cycle counting capacity of $6.5.10^4$. The number of cycles during measurement was totalled. The counter, which formed part of the chronometer, consisted of a decadic electron tube counter with a cycle capacity of 4.10^6, fitted with a mechanical counter for recording the number of cycles (seconds).

2. Basic results of the theory

The experimental results obtained in the present work were compared directly with calculations made on the basis of previous papers [3-7]. It would therefore be convenient to give a brief account of these calculations.

The behavior of a self-oscillator with a loop in the grid-circuit, acted on by an harmonic synchronizing force and subjected to stationary normal fluctuation $\zeta(t)$, can be described, according to [3], by means of a system of "shortened" equations:

$$\dot{A} = \delta\left(1 - \frac{A^2}{A_0^2}\right)A + \frac{\omega E}{2}\cos\varphi + \frac{1}{\lambda A} + \xi_1(t);$$
$$\dot{\varphi} = \Delta - \frac{\omega E}{2A}\sin\varphi + \frac{\xi_2(t)}{A}. \qquad (1)$$

This system is valid on condition that

$$\delta/\omega \ll 1;\ E/2A_0 \ll 1;\ \overline{\left[\int_0^T \omega\zeta(t)\cos(\omega t + \varphi)\,dt\right]^2} \ll A_0^2;$$
$$\tau_{кор} \ll 2A_0/\omega E;\ \tau_{кор} \ll 1/\delta. \qquad (2)$$

Here A and φ are the amplitude and phase of the self-oscillations, A_0 is the amplitude of the unperturbed self-oscillations, $\delta = \omega_0^2(MS - RC)/2$ is the linear part of the system increment, $\Delta \simeq \omega_0 - \omega$ is the amount of mistuning, $1/\lambda = \omega^2\varkappa(\omega)/4$ where

$\varkappa(\omega) = \int_{-\infty}^{\infty}\overline{\zeta_\tau}\cos\omega\tau\,d\tau$ is the spectral density of the fluctuation force and $\xi_1(t)$ and $\xi_2(t)$ are two auxiliary independent normal random functions with a zero mean value, having the correlation function $\overline{\xi_1\xi_{1\tau}} = \overline{\xi_2\xi_{2\tau}} = 2\delta(\tau)/\lambda$

Imposing additional conditions

$$E \ll 2A_0\delta/\omega; \qquad (3)$$
$$\varkappa(\omega) \ll \delta A_0^2/\omega^2 \qquad (4)$$

on the magnitude of the synchronizing force and the fluctuational effect makes it possible to simplify system (1) still further, by substituting A_0 for A in the phase equation, and by solving the corresponding Einstein-Fokker equations to get the laws of distribution of phase and amplitude of the self-oscillations. The law of phase distribution then has the form (see [3]):

$$w(\varphi)=\frac{\exp\{D\varphi+D_0\cos\varphi\}}{4\pi^2 e^{-\pi D}\,|I_{iD}(D_0)|^2}\int_{\varphi}^{\varphi+2\pi}\exp\{-D\psi-D_0\cos\psi\}\,d\psi. \quad (5)$$

Here $D=\lambda A_0^2\Delta$; $D_0=\lambda A_0^2\Delta_0$, where $\Delta_0=\omega E/2A_0$, and $I_{iD}(D_0)$ is a Bessel function of imaginary order and imaginary argument.

We investigated the dependence (5) for different values of D and D_0 by numerical integration in a "Strela" electronic computer in the computer center of Moscow State University. Fig. 7a shows some of the $w(\varphi)$ curves.

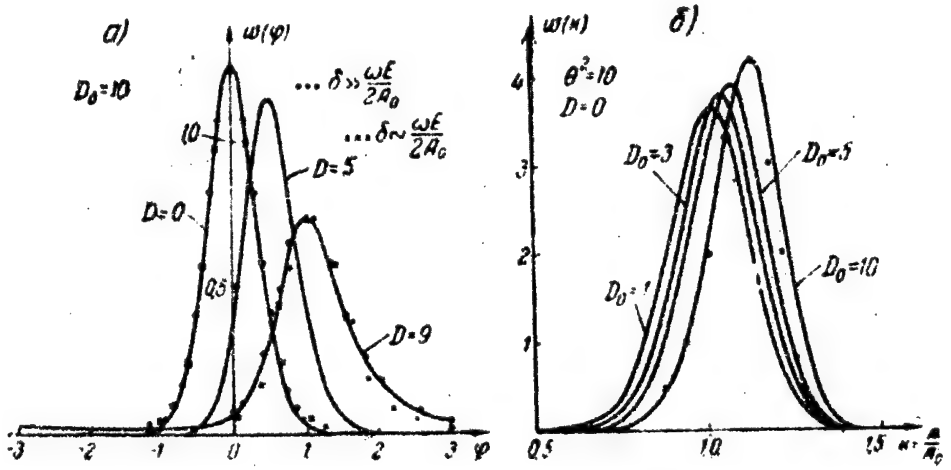


Fig. 7

The law of amplitude distribution for steady-state self-oscillations was found in [3] in the form of a conditional distribution for fixed phase values:

$$w_\varphi(A)=NA\exp\{-\theta^2A_0^{-4}(A^2-A_0^2)^2+\lambda\omega E 2^{-1}A\cos\varphi\}. \quad (6)$$

Here N is the normalization factor, $\theta^2=\delta A_0^2/\omega^2\varkappa(\omega)$. However, experimentally we can only measure an unconditional law of amplitude distribution:

$$w(A)=\int_0^{2\pi}w_\varphi(A)\,w(\varphi)\,d\varphi. \quad (7)$$

We were able to evaluate this expression for the particular case of zero mistuning D = 0. Then

$$w(A/A_0) = N(A/A_0)\ \exp\{-\theta^2[(A/A_0)^2-1]^2\}\, I_0[D_0(1+A/A_0)]. \quad (8)$$

The normalization factor for $D_0 \gg 1$ and $\theta^2 \gg 1$ has the form: $N \approx 4\theta\sqrt{D_0}e^{-2D_0}$. In the remaining cases the normalization factor was determined graphically. Some $w(A/A_0)$ curves are shown in Fig. 7b. If in (5) we neglect the effect of the synchronizing force on the amplitude distribution (which occurs when $\theta^2 \gg D_0$), the distribution law (5) becomes unconditional and assume the simpler form:

$$w(A/A_0) = \frac{4\theta}{A_0\sqrt{\pi}\,(1+\operatorname{erf}\theta)}\,(A/A_0)\exp\{-\theta^2[(A/A_0)^2-1]^2\}. \quad (9)$$

Here erf θ is the error integral.

Disruptions of synchronization, due to the effect of noise and expressed as phase jumps by a whole number of periods, lead to a displacement of the mean frequency of the oscillations of the self-oscillator relative to the frequency of the synchronizing force of magnitude

$$\omega_{cp} - \omega = \Delta\,\frac{\operatorname{sh}(\pi D)}{\pi D}\,|I_{iD}(D_0)|^{-2}. \quad (10)$$

Owing to there being no tables of Bessel functions of imaginary order and imaginary argument in the Soviet literature, only the asymptotic of expression (10) was investigated in [3]. Using tables [11] enabled us to investigate the behavior of the function (10) in detail. Fig. 8 shows the dependence of the frequency displacement on the quantity D, proportional to the mistuning, for different signal-to-noise ratios D_0.

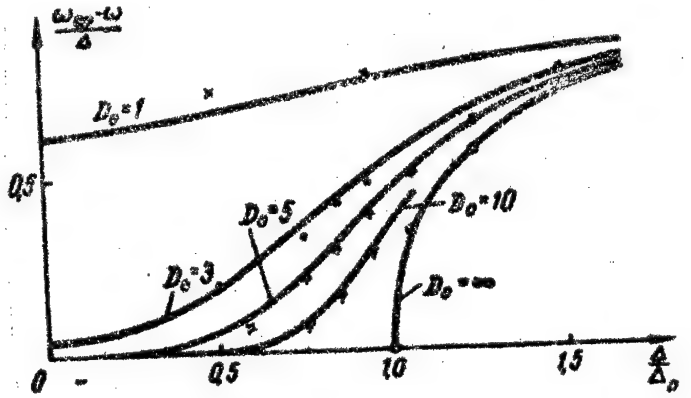


Fig. 8

Not having the opportunity to discuss the transient processes in a self-oscillator in any greater detail, we refer the reader to papers [4-7], in which the results are reduced to formulas and graphs.

3. The experimental results: discussion

Our problem was to investigate the behavior of a synchronized tube oscillator subject to fluctuation noise and, comparing the results with theory, to determine the limits of applicability of the theory evolved in [2-6], which was based on certain simplifying hypotheses. We shall now present the fundamental results of the experiments.

As the experiments showed, the stationary phase fluctuations in an oscillator with a relatively large signal-to-noise ratio ($D_0 \gg 1$) and a small amount of mistuning ($D \ll D_0$) have an almost normal law of distribution, when condition (4) is fulfilled. With a reduction in the signal-to-noise ratio and much mistuning ($D \sim D_0$) the distribution law becomes more and more distorted, deviating from the normal. Comparing the experimental results with expression (5) revealed good agreement, not only when conditions (3) and (4) were fulfilled but also when they were seriously infringed. In the latter case the agreement was better, when we used the steady-state synchronous amplitude for a noiseless oscillator instead of the unperturbed amplitude A_0 in the formulas for D and D_0. The oscillogram in Fig. 9 shows stationary phase distribution curves for $D_0 = 10$, $D = 0$ (Fig. 9a) and for $D_0 = 5$, $D = 4.5$ (Fig. 9b). Condition (3) was observed during the experiment. With the aid of a scaled grid theoretical points for $w^*(\varphi)$ are plotted on the oscillograms. Note that these oscillograms illustrate the method of processing the experimental results. This consisted in printing a calibrated grid scale on the oscillogram negative and then plotting the theoretical points on the grid thus produced on the positive. The two groups of experimental points, plotted on Fig. 7a, which were obtained for observance and non-observance of condition (3), show how little the degree of agreement between the results changes even for a synchronizing force of considerable magnitude.

At low fluctuation levels ($\theta^2 \gg 1$) the stationary amplitude fluctuations in a self-oscillator are also distributed in accordance with an almost Gaussian law. A rise in the noise level leads to the degradation and deformation of the distribution law. A comparison between experimental results and calculated curves (9) shows that the agreement is substantially worse than in the case of phase fluctuations and is satisfactory only for small values of the increment δ. With increase in δ (in the experiment the regeneration was varied) the curves diverge strongly in the region of large amplitudes. This difference may be due to the effect of the control grid currents, which is the stronger the greater the amplitude of the

self-oscillations.

For small amounts of mistuning a small synchronizing signal produced only a slight deformation of the law of amplitude distribution and the degree of agreement with the calculated values also depended essentially on δ . Divergence was noted when the amplitude of the oscillations became relatively large. This could have been the effect of increases in δ or E . In Fig. 7b we have plotted experimental points obtained for small increases in δ and E . As will be seen from the figure, the agreement is satisfactory. When the synchronizing force was subject to a considerable amount of mistuning, the character of the deformation of the distribution law was various, depending on the relationship between the synchronizing signal .... (omission in manuscript), cutting in the synchronizing force led not to the usual decrease but to an increase in the dispersion of the amplitude fluctuations. This agrees with corresponding results in [5].

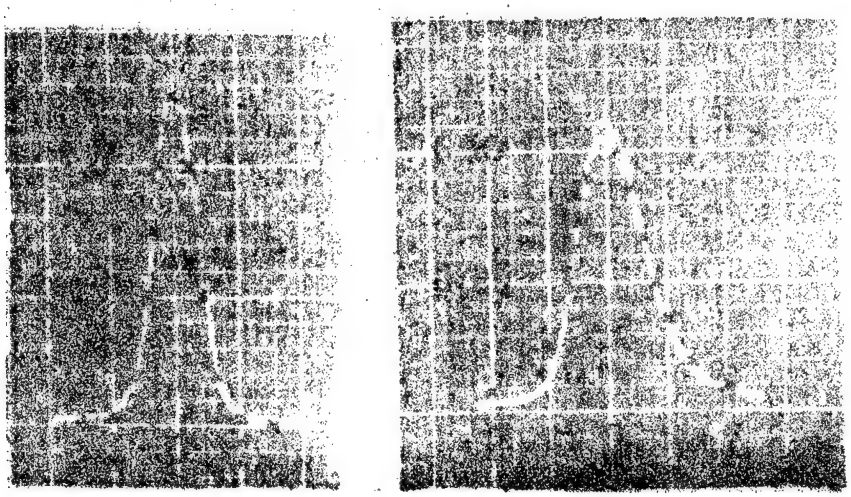


Fig. 9

The experimental investigation of the displacement of the mean frequency of a self-oscillator subjected to the action of noise confirmed the existence of such a displacement and (as with the law of phase distribution) gave good quantitative agreement with formula (10), while upsetting condition (3) only slightly affected agreement with calculations. In passing, with the noise effect cut out, we measured the very fine effect of a partial increase in frequency outside the synchronizing band. The results of frequency measurements are given in Fig. 8.

We now proceed to the experimental results obtained from a study of the processes of establishing the phase and amplitude in a self-

oscillator synchronized by means of radio pulses. Here we must define the boundaries of the "fluctuational" and "non-fluctuational" problems. By the latter we understand the investigation of the probability characteristics of the establishment of synchronous phase and amplitude in a self-oscillator with random initial phase of the synchronizing signal. This effect is a fundamental one in real systems and hence its study is a matter of great interest for practical work. Fluctuation effects deform the results of the "non-fluctuational" problem in a definite manner. The non-stationary laws of phase and amplitude distribution were investigated for both cases.

In the absence of fluctuation effects the character of the variation in the law of phase distribution $w(\varphi,t)$ depends essentially on condition (3) being fulfilled. For small values of the synchronizing signal and slight mistuning $\Delta \ll \Delta_0$ we observed a smooth deformation of the law of phase distribution from uniform distribution with respect to phase at the initial moment to deltoid distribution after a sufficiently long interval of time. This was in excellent agreement with the results of [4,6,7]. When condition (3) was not observed, the character of the changes in the distribution law underwent an essential change. In the intermediate stages the distribution had a "table-shaped" character with sharply receding edges. This qualitative difference from the results of [4,6] was nevertheless in agreement with the results of [7].

Cutting in the fluctuation noise led to the smoothing and erosion of the law of phase distribution. Thus, with a comparatively low level of random interference ($D_0 \leqslant 10$) the qualitative difference between cases of observance and non-observance of condition (3), noted in the absence of fluctuation noise, disappeared. For both cases the non-stationary laws of phase distribution practically coincided and for slight mistuning agreed satisfactorily with the results of approximate calculations of the effect of noise obtained in [4].

Analogous measurements for the process of establishing the amplitude showed that agreement with the results of [5] were only satisfactory at very small amplitudes of the self-oscillations, as long as the grid currents did not play a part.

A characteristic feature of the calculations, used for comparison with the experimental results, is the presence of certain basic simplifying limitations, imposed on the synchronizing and random forces, which made it possible to overcome the mathematical difficulties in the way of a solution of equation (1). Comparing the calculations mentioned with the present experimental results provides an opportunity of extending their sphere of action, since in a number of cases the important infringement of the initial assumptions is due only to noise and the amplitude of the unperturbed self-oscillations. For sufficiently large E and small δ at the edges of the synchronizing band the effect on the true result is inconsiderable. This indicates that the limitations, introduced to permit a solution of problem (1),

are at the same time not especially essential for a number of cases, in the sense that not only the qualitative but also the quantitative aspect is very little affected even if they are seriously infringed. Actually, the steady-state phase fluctuations and the displacement of the mean frequencies of the oscillations due to fluctuation noise are in good agreement with calculated results, even if condition (3) is largely unobserved. The agreement between theoretical and experimental results is not quite so good in determining amplitude distribution, since here the effect of the grid currents, neglected in the theory, assumes importance. However, the agreement improves for weakly regenerated systems. The theory [4-6] correctly describes the process of establishing phase and amplitude in a synchronized self-oscillator in the absence of noise only for sufficiently small amplitudes of the synchronizing signal (condition (3) must be fulfilled). For larger external forces it is necessary to use the more accurate results of [7]. In the presence of sufficiently powerful fluctuation noise ($D \leq 10^{\circ}$) the results obtained in both cases differ only slightly.

Thus, in the present article the limits of applicability of the results of [2-6] are directly defined. Note that to a large extent the results of the experiments can also be related to [1] insofar as the results of [1] and [2-5] not only do not contradict but supplement each other, coinciding at the points of contact.

In conclusion the author expresses his deep gratitude to V.V. Migulin, who suggested the subject and guided it to completion.

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WAVEGUIDE PROPERTIES OF A MULTI-ROW INTERDIGITAL ASSEMBLY

Pages 420-423

by V.N. Ivanov.

The dispersion equation is derived for a multi-row interdigital assembly and solved by the method of successive approximations. It is found that on changing from a plate to an interdigital assembly it is possible to increase considerably the effectiveness of the interaction between an electron flux and the field of a decelerating system. This produces only slight changes in the dispersion properties and thermal dissipation of the system.

The plate assembly, widely used in powerful traveling wave tubes on account of its excellent heat-dissipating properties, has the disadvantage that a large part of the energy stored in it is found in the intervals between plates and consequently does not take part in the interactionw with the electron flux [1]. Moreover, the field, interacting with the electron flux, decreases exponentially with distance from the surface of the plate and this seriously limits the dimensions of the electron flux capable of effective interaction with the field of the decelerating system. As it is traced from the surface into the depth of the assembly, the field in the intervals between plates decreases in accordance with a cosine law so that it was natural to attempt to increase the efficiency of plate assemblies by cutting a series of slots in them and passing the electron flux through these slots in separate streams. The decelerating system thus obtained, consisting of several rows of pins, attached to a conducting support, is called a multi-row interdigital assembly.

1. Dispersion equations

The dispersion properties of a multi-row interdigital assembly may be analyzed by replacing each row of pins with an infinitely thin plate with ideal conduction in the direction of the pins (along the y - axis) and non-conducting in the direction at right angles (Fig.1).

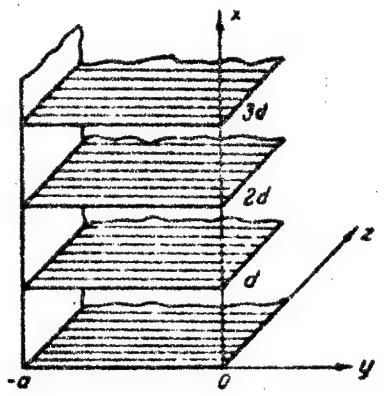

Fig. 1: Multi-row interdigital assembly

We shall also consider the support to be an ideal conductor and take all the components of the field as proportional to the factor $\exp\left[i(\omega t - \beta z)\right]$, which is omitted in what follows.

As current can only flow in the plates in the y-direction, the field can be obtained from the unique component of the vector potential in the y-direction and will not contain the component of the magnetic field along the pins. If we confine ourselves to the case of a system infinite in the x- and z-directions, in which all the variables acquire a multiplier $\exp(-ih_0 d)$ as the result of a shift through one period in the x-direction d, then the components of the field, necessary to a solution of the problem, may be written in the following form: in the region $y < 0$, $0 < x < d$

$$E_y = \sum_{m=1}^{\infty} \left(\beta^2 + \frac{m^2\pi^2}{d^2}\right) A_m \sin\left(\frac{m\pi}{d}x\right) \operatorname{ch}\left[\alpha_m (y+a)\right];$$

$$E_z = ik\beta A\left[(e^{-ih_0 d} - e^{-\beta d})e^{\beta x} + (e^{\beta d} - e^{-ih_0 d})e^{-\beta x}\right] \sin\left[k(y+a)\right] - \quad (1)$$

$$- i\beta \sum_{m=1}^{\infty} \alpha_m A_m \sin\left(\frac{m\pi}{d}x\right) \operatorname{sh}\left[\alpha_m (y+a)\right];$$

$$H_x = -\omega\varepsilon\beta A\left[(e^{-ih_0 d} - e^{-\beta d})\,e^{\beta x} + (e^{\beta d} - e^{-ih_0 d})e^{-\beta x}\right]\cos\left[k(y+a)\right] - \\ - \omega\varepsilon\beta \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi}{d}\,x\right)\operatorname{ch}\left[\alpha_m (y+a)\right], \qquad (1)$$

and in the region $y > 0$

$$E_y = \sum_{n=-\infty}^{+\infty} (h_n^2 + \beta^2) B_n e^{-ih_n x}\, e^{-\gamma_n y};$$

$$E_z = i\beta \sum_{n=-\infty}^{+\infty} \gamma_n B_n e^{-ih_n x}\, e^{-\gamma_n y}; \qquad (2)$$

$$H_x = -\omega\varepsilon\beta \sum_{n=-\infty}^{+\infty} B_n e^{-ih_n x}\, e^{-\gamma_n y},$$

where A, A_m, B_n are constants to be determined, $k = \omega\sqrt{\varepsilon\mu}$, ε and μ are the dielectric constant and permeability of the medium respectively, $\alpha_m^2 = \beta^2 - k^2 + m^2\pi^2 d^{-2}$, $h_n = h_0 + 2\pi n d^{-1}$, $\gamma_n^2 = h_n^2 + \beta^2 - k^2$.

By stipulating the continuity of the components of the field E_y, E_z and H_x in the plane $y = 0$, we get a system of three equations. Eliminating the constant A and then multiplying both the remaining equations by $\sin(m\pi x/d)$, we get (integrating these equations with respect to the variable x between the limits 0 and d) an infinite system of homogeneous equations with respect to the unknown constants B_n. The solubility condition for this system (equating its determinant to zero) gives a dispersion equation, which, after certain simple transformations, can be written in the following form:

$$ka\operatorname{ctg}(ka) = -\Delta_0\left(\sum_{r=-\infty}^{+\infty} \Delta_{1r}\right)^{-1}, \qquad (3)$$

where

$$\Delta_0 = \begin{vmatrix} \ldots M_{1,-1} & M_{1,0} & M_{1,1} \ldots \\ \ldots M_{2,-1} & M_{2,0} & M_{2,1} \ldots \\ \ldots M_{3,-1} & M_{3,0} & M_{3,1} \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix}; \qquad (4)$$

$$M_{m,n} = \frac{\gamma_n a\,(\beta^2 d^2 \pi^{-2} + m^2) + \alpha_m a\,\operatorname{th}(\alpha_m a)\,(h_n^2 d^2 \pi^{-2} + \beta^2 d^2 \pi^{-2})}{h_n^2 a^2 \pi^{-2} - m^2}, \qquad (5)$$

and Δ_{1r} is obtained from the determinant Δ_0 by substituting unity for the elements of the r-th column of this determinant.

In order to solve dispersion equation (3) the infinite determinants are approximated by determinants containing 1x1, 3x3, 5x5, etc. terms, depending on the accuracy required. The difficulties, due to the fact that to calculate values of the quantities $M_{m,n}$ it is necessary to know the value of k (the geometry of the system and the propagation constants h_0 and β are assumed given), are overcome by adopting the method of successive approximations, the null approximation for k being the value determined from the equation

$$ka = (h_0^2 + \beta^2)^{1/2} \cos(k a). \qquad (6)$$

The latter is obtained by equating in the plane y = 0 the impedances (in the y-direction) of the fundamental spatial harmonic in the region $y > 0$ and of the TEM wave in the region $y < 0$.

In the case of cophasal excitation of the interdigital assembly ($h_0 = 0$) the form of the determinant Δ_0 changes: only elements with $n \geqslant 0$ and odd m remain.

2. Calculation of dispersion curves and comparison with experiment

A numerical solution of the dispersion equation was obtained for the case of cophasal excitation, when the width of the system passband is at a maximum. The results of this solution are set out in Fig. 2, in the form of a graph showing k/β, the ratio of the phase velocity of the wave in the z-direction to the velocity of light in a vacuum, as a function of the quantity ka for d/a = 0.3; 0.1 and 0. When d/a = 0.3; 0.1, 3x3 elements were kept in the determinants. The difference between values of k, obtained in this case and obtained with 2x2 elements left in the determinants, was less than 1 %. In solving the dispersion equation the successive approximations converge so quickly that it is possible to confine oneself to the first approximation.

The graph also illustrates a solution of the dispersion equation for a single-row assembly ($d/a = \infty$), which, if the deceleration is considerable, may be approximately written in the form [2]:

$$k/\beta = \sin(2 ka). \qquad (7)$$

As is evident from the graph, the dispersion curves for a multirow interdigital assembly lie between the dispersion curve for a plate assembly, obtained by neglecting the periodicity of this system (curve d/a = 0) [3], and the curve for a single-row interdigital assembly.

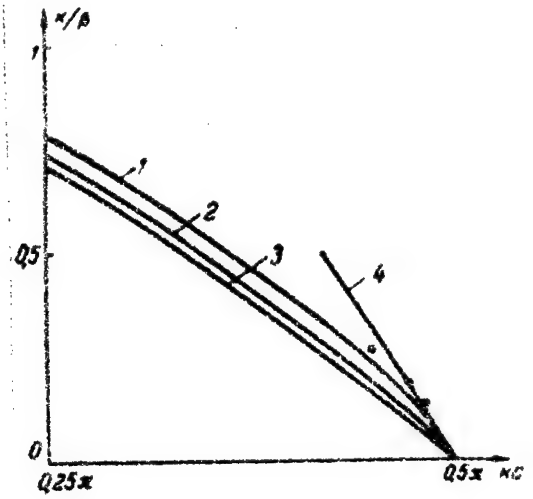

Fig. 2: Dispersion curves for the case of cophasal excitation.

1) Multi-row digital assembly (MDA) for d/a = 0.3
2) MDA for d/a = 0.1
3) Plate assembly and MDA for d/a = 0
4) Single-row interdigital assembly (d/a = ∞)

o - Experimental results for a five-row interdigital assembly for d/a = 0.3.

For small values of the ratio d/a the dispersion curves for a multi-row interdigital assembly and a plate assembly approach each other.

The results plotted refer to the experimental investigation of a five-row interdigital assembly, with the following dimensions: d/a = 0.3, length of pin = 10 cm, diameter of pin 3 mm, distance between pins in the same row 2 cm. Small discrepancies between the calculated (curve 1) and measured values of the phase velocity may be due to the fact that the theory outlined above does not take into account the periodicity of the system along the z - axis.

The results of analysis of the dispersion properties of a multi-row interdigital assembly show that the change from a plate to an interdigital assembly leads to a considerable increase in the dimensions of the electron flux effectively interacting with the electromagnetic field of the decelerating system, without materially affecting its dispersion properties or its capacity to dissipate heat. Consequently, such an exchange would increase the amplification factor and the output

power of traveling wave tubes with decelerating systems employing this type of assembly.

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ON THE EFFECT OF RANDOM ERRORS IN SOURCE DISTRIBUTION ON THE RADIATION PATTERNS OF TRAVELING WAVE ANTENNAS.

Pages 424-430

by V.I. Talanov & N.M. Sheronova.

The effect of random errors in source distribution on the radiation patterns of traveling wave antennas is examined. Expressions are obtained for the deviations of the average radiation pattern from the nominal due to random perturbations of the amplitude, phase and phase velocity of the current wave in the antenna. It is shown that random perturbations of the phase velocity limit the possibility of getting high directivity factors at the expense of an increase in the size of the antenna.

1. In using high-directional antennas of the traveling wave type (leaky waveguide, surface-wave, etc.) for radar and radioastronomical purposes we encounter the question of the effect of the accuracy with which the various elements of such antennas are made on the radiation patterns of these devices. The authors of [1-5] have discussed the analogous problem of the effect of random errors in source distribution on the radiation patterns of multivibrator (multi-slot) antennas and of lens and reflector antennas also.

In the case of a traveling wave antenna the errors in source distribution may be due to inaccuracy in the fabrication of the radiating elements themselves or of the waveguide channel feeding high-frequency energy to these elements, while as on account of the coupling between the radiator and the transmission line these inaccuracies are equivalent (in the sense of the effect on the propagation of the wave in the line) to irregularities in the line itself. From the point of view of distortion of the radiation pattern in the

presence of random perturbations the characteristic feature of traveling wave antennas is the fact that irregularity of the transmission line (and of the radiating elements also), even over a small section, affects the distribution of the currents, generally speaking, over the whole antenna. Thus, a variation in the phase velocity of a wave in the line over an individual finite section of the latter leads to misphasing of parts of the antenna lying on different sides of this section. Thus, if as a first approximation we can consider that for lens and reflector antennas a change in the profile of the reflector or the lens over a given section leads to distortion in the source distribution only over the same section, then for a traveling wave antenna random perturbations of a given source distribution have, generally speaking, a non-local character. Below we use the example of a linear antenna with continuous distribution of the current to illustrate some characteristics of the effect of different kinds of random error (amplitude, local and non-local phase) on the radiation pattern of a traveling wave antenna.

2. Assuming that the antenna parameters are slowly varying functions of the coordinate z , we shall characterize the source distribution by the function:

$$I(z) = I_0(z)\, e^{-i\psi(z)}, \qquad (1)$$

where the real functions $I_0(z)$ and $\psi(z)$ describe the distribution of the amplitudes and phases of the currents in the antenna respectively.

The (power) radiation pattern of an antenna of length 2L with a current distribution (1) is described correct to a constant factor by the function (without taking into account the radiation pattern for the individual element of current):

$$\Phi(u) = \int_{-L}^{L}\int_{-L}^{L} I(z)\, I^*(z)\, e^{ik u (z-z')}\, dz\, dz', \qquad (2)$$

where $u = \cos\theta$ (θ is the angle between the antenna axis and the direction to the observation point), $k = 2\pi/\lambda_0$, λ_0 is the wavelength in free space.

In the presence of random errors in the current distribution:

$$\Delta I(z) = I(z) - \langle I(z) \rangle$$

for the average (with respect to an ensemble of antennas of the same type) radiation pattern $\langle \Phi(u) \rangle$ we get the following expression:

$$\langle \Phi(u) \rangle = \Phi_0(u) + \int_{-L}^{L}\int_{-L}^{L} K(z, z') e^{ihu(z-z')}\, dz\, dz', \tag{3}$$

where

$$\Phi_0(u) = \int_{-L}^{L}\int_{-L}^{L} \langle I(z) \rangle \langle I^*(z') \rangle e^{ihu(z-z')}\, dz\, dz' \tag{4}$$

is the radiation pattern for a nominal antenna with a current distribution $\langle I(z) \rangle$;

$$K(z, z') = \langle I(z) I^*(z') \rangle - \langle I(z) \rangle \langle I^*(z) \rangle . \tag{5}$$

Following [1], we shall call the second term in (3)

$$\Phi_{pac} = \int_{-L}^{L}\int_{-L}^{L} K(z, z')\, e^{ihu(z-z')}\, dz\, dz' = \langle \Phi(u) \rangle - \Phi_0(u) \tag{6}$$

the dissipated power.

3. We shall consider that the dimensions of the inhomogeneities in the transmission line, supplying energy to the radiating elements, are large compared with the length of the wave in the line and put the phase of the current (1) $\psi(z)$ in the form:

$$\psi(z) = \psi_0(z) + \int_0^z h(z')\, dz' . \tag{7}$$

Here the first term $\psi_0(z)$ may be interpreted as the phase difference between the field of the wave in the line and the current in the radiating element for the same value of the coordinate z .

Random deviations from the mean $\Delta I(z)$ in the current distribution for a given antenna may be due both to amplitude

$$\Delta I_0(z) = I_0(z) - \langle I_0(z) \rangle , \tag{8}$$

and phase errors

$$\Delta\psi(z) = \psi(z) - \langle \psi(z) \rangle , \tag{9}$$

while in the latter case we shall somewhat arbitrarily distinguish

a) local phase errors

$$\Delta\psi_0(z) = \psi_0(z) - \langle \psi_0(z) \rangle , \tag{10}$$

connected with random deviations from the mean of parameters of a given radiating element and

b) non-local phase errors due to random fluctuations in the propagation constant

$$\Delta h(z) = h(z) - \langle h(z) \rangle . \tag{11}$$

Henceforth, for the sake of simplicity, we shall assume that $\langle \psi_0(z) \rangle = 0$ and $\langle h(z) \rangle = \text{const} = h_0$.

We shall consider the effect of each kind of error on the radiation pattern of the antenna, assuming that the random quantities $\Delta\psi_0$ ΔI_0 and Δh are not correlated with each other*; moreover we shall confine ourselves to computing the dissipated power (6), characterizing, on the average, deviations of the radiation pattern of the antenna from the nominal. In order to take into account correlation of values of the random functions $\Delta I_0(z)$, $\Delta\psi_0(z)$ and $\Delta h(z)$ at different points z, z', we shall take:

$$K_{I_0}(z, z') \equiv \langle \Delta I_0(z)\, \Delta I_0(z') \rangle = \langle \Delta I_0^2 \rangle \exp\left(-\frac{|z - z'|}{C_{I_0}}\right); \tag{12}$$

$$K_{\psi_0}(z, z') \equiv \langle \Delta\psi_0(z)\, \Delta\psi_0(z') \rangle = \langle \Delta\psi_0^2 \rangle \exp\left(-\frac{|z - z'|}{C_{\psi_0}}\right); \tag{13}$$

$$K_h(z, z') \equiv \langle \Delta h(z)\, \Delta h(z') \rangle = \langle \Delta h^2 \rangle \exp\left(-\frac{|z - z'|}{C_h}\right), \tag{14}$$

where C_{I_0}, C_{ψ_0}, C_h are the correlation radii of random deviations in amplitude, phase ψ_0 and propagation constant h respectively. We shall assume that C_{I_0}, C_{ψ_0}, $C_h \ll L$ and that the mean value of the current amplitude is subject only to small variations at distances of the order of the correlation radii C_{I_0}, C_{ψ_0}, C_h.

* Although such an assumption is far from being always fulfilled, it still permits an evaluation of the distortions in an antenna radiation pattern, introduced by errors of different kinds.

4. By expressing the function K(z,z') (5) in terms of $K_{Io}(z,z')$:

$$K(z, z') = K_{I_0}(z, z') \exp\left| -ih_0 (z - z') \right|$$

and substituting this value in (6), we find that in the case of amplitude errors the dissipated power $\Phi_{pac}(u)$ is equal to

$$\Phi_{pac}(u) = \Phi_{pac}(\xi) = 4L^2 f(t, \xi) \tilde{D}_{I_0}, \qquad (15)$$

where

$$f(t, \xi) = \frac{1}{t(1 + \xi^2/t^2)};$$

$$t = L/C_{I_0}; \quad \xi = (ku - h_0) L; \quad \tilde{D}_{I_0} = \frac{1}{2L} \int_{-L}^{L} D_{I_0}(z)\, dz .$$

In deriving (15) we assumed that at distances of the order of C_{Io} the dispersion $D_{Io} = \langle \Delta I_o^2(z) \rangle$ varies only slightly.

From the graph for $f(\xi, t)$, shown in Fig. 1, it is evident that for a fixed antenna length 2L and a given value of the mean dispersion $\tilde{D}_{Io}$ the dissipated power level and also the directivity of the diagram $\Phi_{pac}(\xi)$ increase with increase in the correlation radius C_{Io}. The direction of the maximum of $\Phi_{pac}(\xi)$ coincides with the direction of the principal maximum $\xi = 0$ of the $\Phi_o(\xi)$ diagram of the nominal antenna with symmetrical amplitude distribution $I_o(z) = I_o(-z)$. The relative distortions of the diagram, which can be characterized by the ratio $\Phi_{pac}(\xi)/\Phi_o(\xi)$, decrease with increase in the size of the antenna as 1/L, if the magnitude of the errors and also the correlation radius C_{Io} remain constant.

In the case of local phase errors (10), assuming the errors to be small, we find that

$$K(z, z') = 2^{-1} I_0(z) I_0(z') \exp\left| -ih_0 (z - z') \right| \langle \left| \Delta\psi_0(z) - \Delta\psi_0(z') \right|^2 \rangle .$$

Substituting this expression in (6) and taking into account (13), we gat the following expression for the dissipated power:

$$\Phi_{pac} = -\langle \Delta\psi_0^2 \rangle \Phi_0 + 4 \langle \Delta\psi_0^2 \rangle L^2 f(t, \xi) \tilde{I}_0^2, \qquad (16)$$

where

$$\tilde{P}_0^2 = \frac{1}{2L}\int_{-L}^{L} P_0^2(z)\,dz; \quad t = \frac{L}{C_{\varphi_0}}.$$

Here the dissipated power is composed of two terms; the first is proportional to the nominal diagram, the second coincides, correct to a constant multiplier, with $\Phi_{\text{рас}}(\xi)$ for amplitude errors.

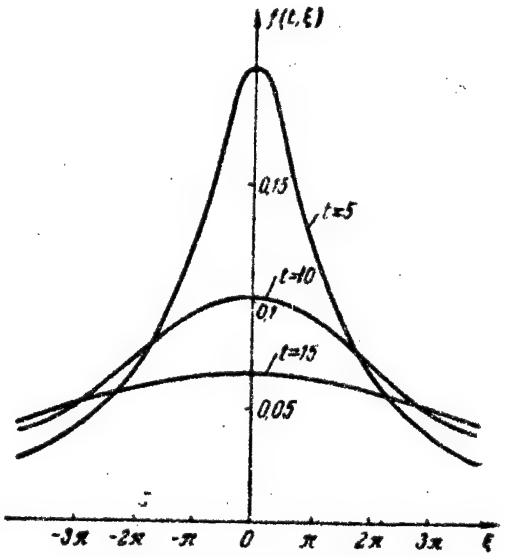

Fig. 1

5. The results presented above are completely analogous to the findings in [2,3] for the case of antennas with circular and rectangular apertures. The non-local phase errors (11) are specific for traveling wave antennas. Assuming that

$$\left|\int_z^{z'} \Delta h(z'')\,dz''\right| \ll 1,$$

we have

(17)

$$K(z,z') = \frac{1}{2} I_0(z) I_0(z') \exp\left[-ih_0(z-z')\right] \left\langle \left| \int_z^{z'} \Delta h(z'')\,dz'' \right|^2 \right\rangle,$$

where, in accordance with (14),:

$$\left\langle \left| \int_{z}^{z'} \Delta h(z'')dz'' \right|^2 \right\rangle = 2\left\{ C_h \langle \Delta h^2 \rangle |z-z'| + C_h^2 \langle \Delta h^2 \rangle \times \right.$$
$$\left. \times \left[\exp\left(-\frac{|z-z'|}{C_h} \right) - 1 \right] \right\}. \qquad (18)$$

Thus, even for a small $\langle \Delta h^2 \rangle$ the mean square error of difference in phase at two points of the antenna may be large, if these points are sufficiently far apart. Therefore, we may anticipate that in this case the character of the depenendence of the dissipated power on the length of the antenna will differ qualitatively from the case of amplitude and local phase errors.

Substituting (17) in (6) and assuming, for simplicity, that $I_0(z) = I_0 = \text{const}$, we get:

$$\Phi_0(u) \equiv \Phi_0(\xi) = 4L^2 I_0^2 f_0(\xi) \equiv 4L^2 I_0^2 \frac{\sin^2\xi}{\xi^2}; \qquad (19)$$

where $$\Phi_{pac}(u) \equiv \Phi_{pac}(\xi) = 4L^4 I_0^2 \langle \Delta h^2 \rangle \tilde{f}(t, \xi), \qquad (20)$$

$$\tilde{f}(t, \xi) = \frac{1}{t^3} \frac{1}{1 + \xi^2/t^2} + \frac{1}{t^2} \frac{\sin^2\xi}{\xi^2} - \frac{\sin 2\xi - 2\xi\cos^2\xi}{t\,\xi^3}. \qquad (21)$$

Bearing in mind that $t = L/C \gg 1$, it is easy to see that the main role in (21) is played by the last addend

$$f_1 = -(\sin 2\xi - 2\xi\cos^2\xi)/t\,\xi^3.$$

As far as the first two terms are concerned, they are analogous to the expressions defining the dissipated power in the case of amplitude and local phase errors. Fig. 2 presents graphs of the functions $f_0(\xi)$ and $f_1(\xi)$ for different values of t. The function $\tilde{f}(\xi)$ and, consequently, the diagram of $\Phi_{pac}(\xi)$ (20) have a sharp maximum, coinciding in direction with the maximum of the nominal diagram but having a different sign. This means that on the average the level of radiation in this direction is significantly reduced for sufficiently large errors $\langle \Delta h^2 \rangle$.

Now, as distinct from the case of amplitude and local phase

errors, the level of $\Phi_{pac}(\xi)$ at the maximum of the diagram ($\xi = 0$) is proportional to L^3 and, consequently, increases with increase in L more rapidly than the level of the diagram of the nominal antenna, which is proportional to the square of the length. Hence it is clear that even when the mean square error in the phase constant $\langle \Delta h^2 \rangle$ does not vary with increase in L, there is a certain limiting antenna length, depending on $\langle \Delta h^2 \rangle$ and C_h, to exceed which with the intent of getting greater directivity is inexpedient. To evaluate this limiting length we assume that in the presence of errors the total power radiated by the antenna remains approximately equal to the power radiated by the nominal antenna. Then for a directive gain G, calculated from the average diagram $\langle \Phi(\xi) \rangle$, in the direction of the maximum of the nominal diagram we have:

$$G \approx \beta L \left(1 - \frac{2}{3} \langle \Delta h^2 \rangle L C_h\right), \qquad (22)$$

where β is a constant factor not depending on the length of the antenna.

By the limiting antenna length L_{npeD} we shall understand the value of L, at which G(L) has a maximum:

$$\frac{L_{npeD}}{\lambda_c} \approx \frac{3}{16\pi^2} \frac{1}{(\langle \Delta h^2 \rangle / h_0^2)(C_h/\lambda_c)},$$

where $\lambda_c = 2\pi/h_o$ is the length of the wave in the line. The maximum value of the directive gain

$$G_{max} = 1/2 . G_o(L_{npeD})$$

proves to be equal to half the directive gain for the nominal antenna for the same length L_{npeD}.

For example, for a standard 3 cm waveguide with a tolerance of $\pm$ 0.05 mm for the dimensions of the internal cross section and $C_h/\lambda_c \sim 10$ we find that $2L_{npeD} \sim 35$ m. Actually, owing to other factors, not taken into account here, and also owing to a small variation in G with increase in L close to $L = L_{npeD}$, the length of the antenna ought to be taken as smaller than the limiting value obtained

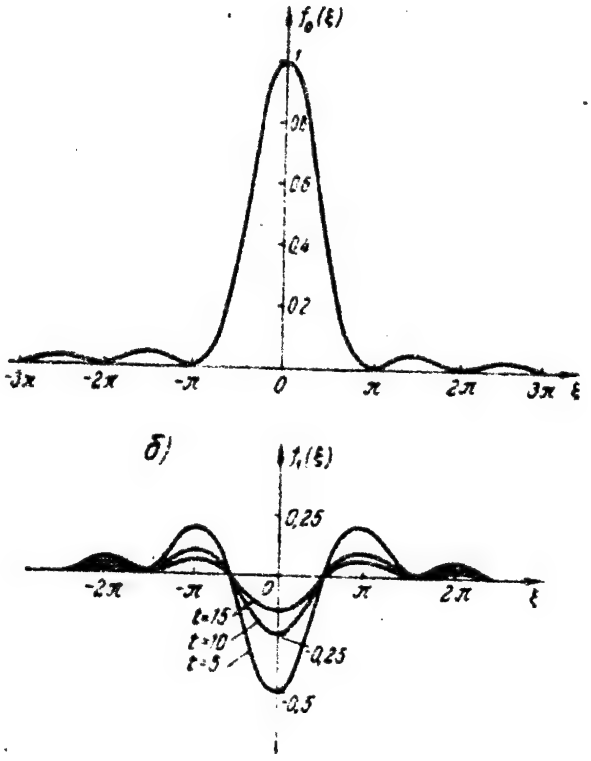

Fig. 2

here. For an increase in the allowable limiting length of the antenna the accuracy, with which the waveguide channel is fabricated, ought to be increased; for the same tolerances for an increase in $L_{пред}$ it is desirable to have a smaller value of C_n/λ_c.

The effect of random errors on the radiation patterns of traveling wave antennas with continuous source distribution shows that in such antennas the most important role is played by phase errors of a non-local character, due to the presence of a connection between the individual radiating elements via the field of the wave in the transmission line forming part of the antenna. The relative distortions, introduced by these errors, are proportional, on the average, to the length of the antenna which limits the possibility of getting sufficiently high directive gains at the expense of an increase in the size of the antenna.

An analogous picture will be found for leaky waveguide antennas with a discrete arrangement of the radiating elements. In this case the effect of random perturbations of the parameters of individual radiating elements on the phase relationships between them may be calculated on the basis of a solution of the problem of the propagation of waves in a system of series-connected multipoles with random variation of the parameters [6,7].

Above we confined ourselves to a consideration of the dissipated power Φ_{pac}, characterizing, obviously, the amount of dispersion of the amplitude of the radiation field. As far as the law of distribution of probabilities of random deviations of the diagram from the mean is concerned, making certain assumptions, which permit us to use Lyapunov's theorem, it is possible to reckon that the imaginary and real parts of the amplitude-phase directional diagram with respect to the field are subject to a two-dimensional normal law of distribution [8]. The latter reduces approximately to a unidimensional normal law of distribution of amplitudes of the field for large mean values of these amplitudes (for example, at the maximum of the diagram) and to a Rayleigh law close to the zeroes of the nominal diagram.

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Addendum: The above evaluation of the effect of non-local phase errors is strictly correct only on condition that the random deviations of the difference in phase at any two points on the antenna from the mean are small. If these deviations are not small, then it is possible to get a more general result by assuming that they are distributed in accordance with a normal law with a mean value equal to zero and the dispersion (18). The average diagram is then easy to calculate on the assumption that $\langle \Delta h \rangle \sigma_h^2 \ll 1$. For the directive gain instead of (22)

we shall have

$$G = 4\beta(\langle \Delta h^2 \rangle C_h x)^{-1}(e^{-x} - 1 + x) \quad (x = 2\langle \Delta h^2 \rangle C_h L). \qquad (*)$$

Formula (22) is obtained from (*) if in this expression we take the first four terms of an expansion of the exponential in powers of x . According to (*) with increase in L the directive gain tends monotonically towards the value $G_\infty = 4\beta(\langle \Delta h^2 \rangle C_h)^{-1}$.

When $L = L_{\text{пред}}$ $G \simeq G_\infty/2$; a further increase in the directive gain requires,however, a disproportionately large increase in the length of the antenna, which in practice is scarcely possible.

The authors are grateful to V.Yu. Rudin, whose note pointing out the result (*) made this addendum possible.

GAS DISCHARGE RETUNING OF CAVITY RESONATORS

Pages 431-437

by I.T. Byzova

The results of using a gas discharge for retuning cavity resonators are given. Retuning may be done in two ways: by varying the discharge current and by superimposing on the discharge an external steady magnetic field, close to resonance. The first method gives lower relative losses than the second. Maximum retuning for comparatively small relative losses was obtained with dischargers, the electrodes of which consisted of two thin metal filaments, stretched along the axis of the discharge tube a few millimaters apart.

The electronic retuning of cavity resonators over a wide interval of frequencies has an application to many problems of high-frequency technology. If an ionized gas is introduced into a resonator, then the natural frequency of the resonator will depend on the parameters of the plasma; by varying the parameters of the plasma it is possible to retune the resonator. Such a retuning is inevitably accompanied by an increase in the high-frequency losses, i.e. by a lowering of the Q-factor.

This article deals with an attempt to find the optimum conditions for getting maximum retuning with the aid of a gas discharge, while keeping high-frequency losses to a minimum.

In retuning resonators it is possible to make use of the phenomenon of cyclotron resonance of the plasma electrons [1]. The admittance of a gas-discharge plasma, to which there is applied a steady magnetic field $\underline{H}$, perpendicular to the high-frequency electric field, is expressed as a tensor [2]. The admittance in the direction of the high-frequency electric field $\underline{E}$ takes the form [1]:

$$\sigma_E = \sigma_r + i\sigma_i = N\frac{e^2}{m}\left[\frac{\nu(\nu^2+\omega^2+\omega_H^2)+i\omega(\nu^2+\omega^2-\omega_H^2)}{(\nu^2-\omega^2+\omega_H^2)^2+4\omega^2\nu^2}\right]. \quad (1)$$

Here N is the electron density, e,m the charge and mass of an electron,

ν is the frequency of collisions between an electron and gas molecules and ions, $\omega = 2\pi f$ is the angular frequency of the external high-frequency field, $\omega_H = eH/mc$ is the cyclotron frequency, c is the speed of light.

Plasma retuning of a resonator is mainly governed by the presence of a reactive component of the admittance σ_i. The active component of the admittance σ_r determines the high-frequency losses in the plasma and leads to a lowering of the Q-factor of the resonator. A resonator containing a plasma may be retuned in two ways, firstly by varying the electron density, which can be achieved, for example, by changing the discharge current, secondly, by making use of the phenomenon of the cyclotron frequency of the plasma electrons. In the latter case retuning is accomplished by varying the external steady magnetic field.

In both cases a change in the reactive component of the admittance is accompanied by a change in its active component. We shall compare the relative high-frequency losses for the two methods. For this purpose we find the ratio σ_r/σ_i from expression (1):

$$\frac{\sigma_r}{\sigma_i} = \frac{\nu}{\omega}\,\frac{\nu^2+\omega^2+\omega_H^2}{\nu^2+\omega^2-\omega_H^2}. \qquad (2)$$

It is evident from (2) that in the presence of the magnetic field $\underline{H}$ the ratio σ_r/σ_i is always greater than in the case $\underline{H} = 0$, i.e. applying a magnetic field must lead to an increase in the relative losses.

In certain practical applications it is important it is important to know not only the magnitude of the relative losses but also the rate of variation in the losses on increasing the reactive component of the admittance, i.e. the value of $d\sigma_r/d\sigma_i$.

We shall denote the rate of change in the relative losses in current retuning (when H = 0) by $(d\sigma_r/d\sigma_i)_N$ and the analogous quantity for magnetic field retuning for invariable N by $(d\sigma_r/d\sigma_i)_H$. Comparing these quantities, we find that

$$\left|(d\sigma_r/d\sigma_i)_N\right| < \left|(d\sigma_r/d\sigma_i)_H\right|. \qquad (3)$$

This inequality is obtained for the condition that $\nu/\omega \ll 1$, a condition which is fulfilled for the 10 centimeter waveband for gas pressures of several mm Hg and for weak high-frequency fields [3]. Inequality (3) is fulfilled for the following values of the steady magnetic field:

$$0 < eH < mc(\omega - \nu); \qquad mc(\omega - \nu) < eH < \infty \qquad (4)$$

in the interval:

$$mc(\omega - \nu) < eH < mc(\omega + \nu), \qquad (5)$$

i.e. in the interval of values of the magnetic field between the extremums of the function $\sigma_i(H)$ the losses are very large [1]. Therefore we shall not consider values of $d\sigma_r/d\sigma_i$ and σ_r/σ_i in the interval (5).

Thus, in the case of magnetic field retuning of the resonator the relative losses and the rate of increase in these losses are greater than in the case of discharge current retuning. Since this reasoning does not take into account variations in the temperature of the electron gas and the frequency of collisions between electrons and atoms, the results obtained may prove to be a very rough approximation to reality. Therefore the relative high-frequency losses of the two methods of retuning were compared experimentally.

The measuring setup is shown in Fig. 1. A discharge tube, 27.2 mm in diameter with plane electrodes, was placed in the middle of a resonator, measuring 34x72x436 mm, at right-angles to its length. H_{105} oscillations were excited in the resonator. For a cold discharge tube the Q-factor of the resonator Q_o was equal to 3,500 - 4,000. An external steady magnetic field was directed along the axis of the discharge tube, at right-angles to the high-frequency electric field. The resonance frequency and the Q-factor of the resonator, with and without discharge, were measured by comparing resonance curves [4]. A wavemeter with a Q-factor of 10^4 and graduated in 10^{-1} kc was used for this purpose. With the aid of a nonius it was possible to measure the relative changes in frequency correct to 10^{-2} mc.

Typical curves for resonator retuning Δf and high-frequency losses in the discharge $(1/Q - 1/Q_o)$ (where Q_o is the Q-factor of the resonator with a cold discharger and Q is the Q-factor of the resonator with discharge) are given in figures 2 and 3. The graphs showing the high-frequency losses in discharge as a function of the retuning (Fig. 4) support the conclusions of theory. Discharge current retuning gives lower relative losses than retuning with an external magnetic field.

An analysis of formula (1) shows that for increasing the retuning range of a resonator it is advantageous to get a discharge with as great an electron density N as possible and that in order to decrease the high-frequency losses, introduced by the discharge, the frequency of collisions between electrons and atoms ν ought to be as low as possible. In order to satisfy these conditions, we employed a glow discharge between electrodes consisting of two parallel thin metal filaments. The electrodes were arranged at a distance d = 5 - 7 mm apart along the axis of the tube. The configuration of the constant electric field of this system was such that at a distance exceeding d the intensity of the electric field was small and, consequently, at the periphery of the discharge the electron temperature was low. The low

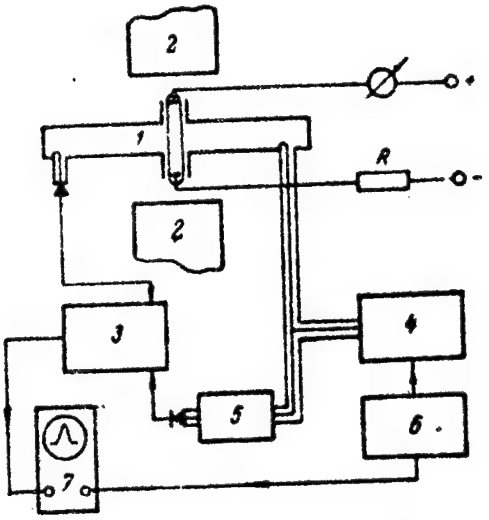

Fig. 1: Block diagram of measuring setup:
1 - measuring resonator with discharge tube; 2 - poles of electromagnet; 3 - amplifier-mixer; 4 - klystron generator; 5 - wavemeter; 6 - saw-toothed voltage generator; 7 - oscillograph.

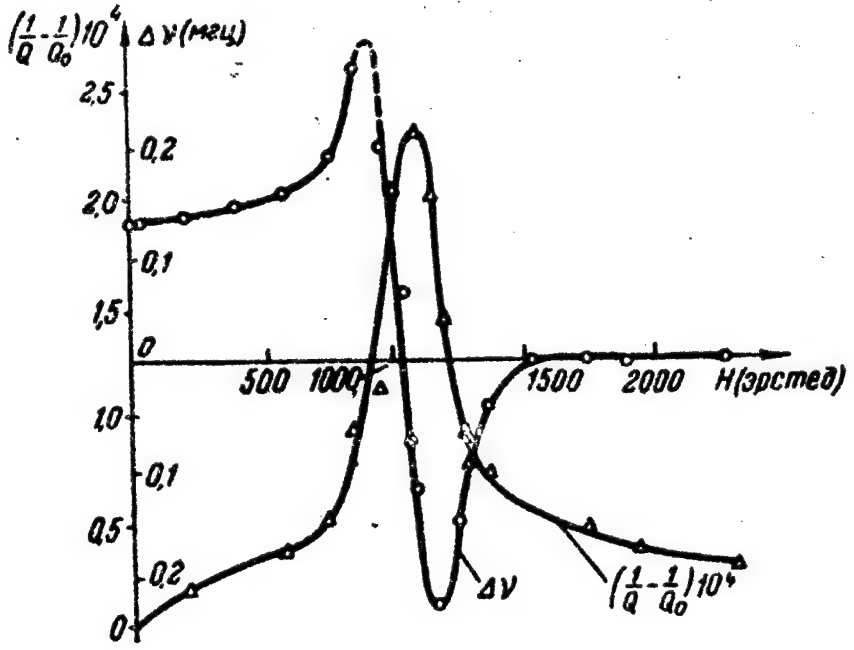

Fig. 2: High-frequency losses in discharge $(1/Q-1/Q_0)$ and resonator retuning Δf as functions of the value of the external steady magnetic field H (Ne 0.95 mm Hg).

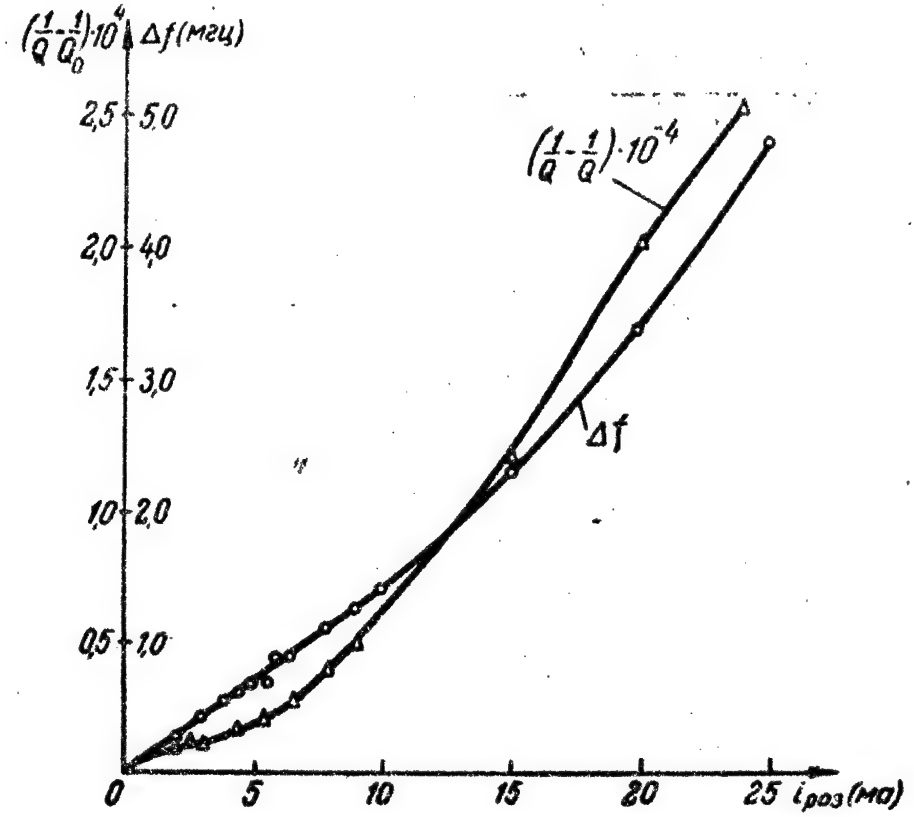

Fig. 3: High-frequency losses in discharge and resonator retuning as functions of the discharge current i_{dis} (Ne 0.95 mm Hg).

temperature of the electron gas leads to a decrease in the high-frequency losses in discharge and to a decrease in the diffusion of electrons from the discharge gap. The diffusion of particles to the electrodes is also made more difficult in that the surface of the electrodes is relatively small. It may be anticipated that in a discharge with such a configuration the electron density will be higher than in a discharge between plane electrodes.

The considerations relating to the advantages of a discharger with two parallel filament electrodes have a purely qualitative character, since the calculations for the system described are very complex. With dischargers of this design we obtained maximum retuning with minimum losses.

The measurements were made in the 10 cm and 3 cm wavebands. The measuring setup was analogous to that shown in Fig. 1. In the 3 cm range we used a cylindrical resonator, 24 mm in diameter, with two filament electrodes stretched along the axis 3 mm apart and insulated

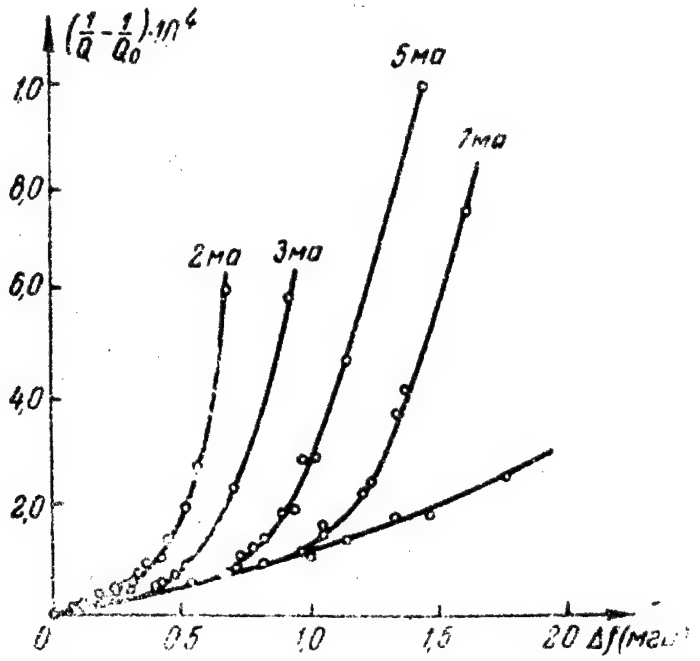

Fig. 4: Connection between high-frequency losses in discharge and change in the resonator resonance frequency for discharge current i_{dis} retuning (lower curve) and magnetic field retuning with fixed discharge current (side branches) (Ne 0.85 mm Hg).

from the body of the device. The discharge was triggered between the filaments; the body of the resonator was connected to one of the electrodes. An H_{118} wave was excited in the resonator. The coupling between the resonator and the klystron and detector head was effected by means of inductive slits in the end faces. Discharge tubes of this design were filled with argon and neon at pressures of from 5.10^{-2} mm Hg to several mm Hg. The maximum frequency retuning of a resonator, filled by a discharge, was of the order of 200 mc for a variation in the Q-factor from 300 to 1,500 (Fig. 5).

In the 10 cm range we used flattened discharge tubes with filaments stretched along the axis. The tubes were arranged in a rectangular resonator of dimensions 34x72x436 mm. H_{107} oscillations were excited in the resonator. The measurements were made using neon and helium in the pressure range from 0.4 to 6 mm Hg. The results of the measurements are shown in Figs. 6 and 7. The relative losses were compared for helium and neon at pressures of 1.7 mm and 2.1 mm. For this purpose we plotted curves showing the dependence of the losses in discharge, i.e.

values of $1/Q - 1/Q_0$ on the retuning of the resonator. The relative losses proved to be less in neon than in helium.

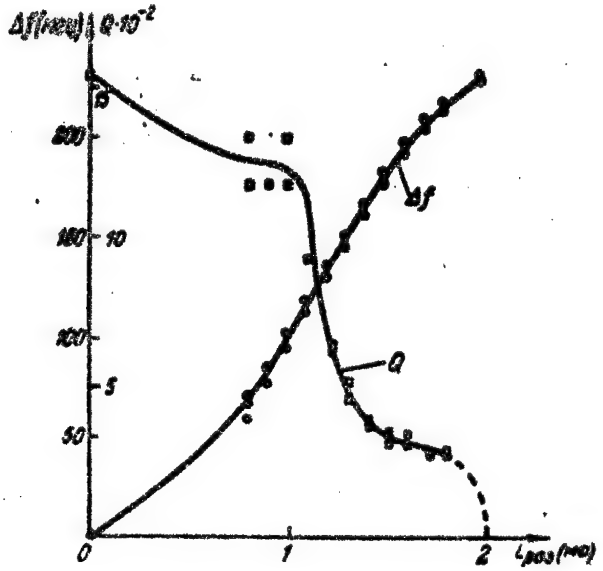

Fig. 5: Retuning and Q-factor of resonator as functions of the discharge current (Ne 5.4 mm Hg, $\Delta f = 0$ corresponds to the frequency f = 9,275 mc).

The results of our investigations may be summarized as follows:

1. Maximum retuning of resonators for minimum losses was obtained with dischargers having electrodes in the form of two thin filaments stretched several mm apart.

2. In the 10-centimeter range the maximum retuning for minimum losses was obtained with dischargers filled with neon at pressures of from 1.5 mm to 2 mm Hg and in the 3-centimeter range with dischargers filled with neon up to pressures of 5.4 mm Hg.

3. Impressing an external steady magnetic field on the discharge increases the relative losses in the high-frequency field.

In conclusion the authors feels obliged to express his gratitude to Yu.V. Gorokhov for his interest in the work.

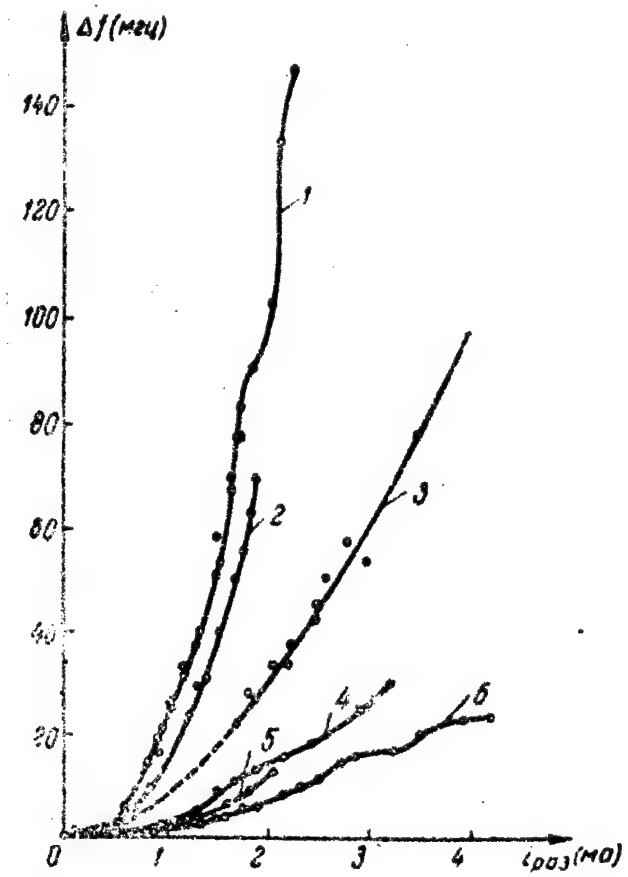

Fig. 6: Resonator retuning as a function of the discharge current in the 10 cm range (neon-filled):

1) pressure p = 1.5 mm Hg, Q_o = 1,200, Q = 130;
2) p = 2.1 mm Hg, Q_o = 850, Q = 490;
3) p = 1.7 mm Hg, Q_o = 560, Q Q = 310;
4) p = 0.7 mm Hg, Q_o = 990 Q = 130;
5) p = 6.1 mm Hg, Q_o = 1,100, Q = 820;
6) p = 4.1 mm Hg, Q_o = 660, Q = 220.

The Q-factor Q is given for maximum discharge current.

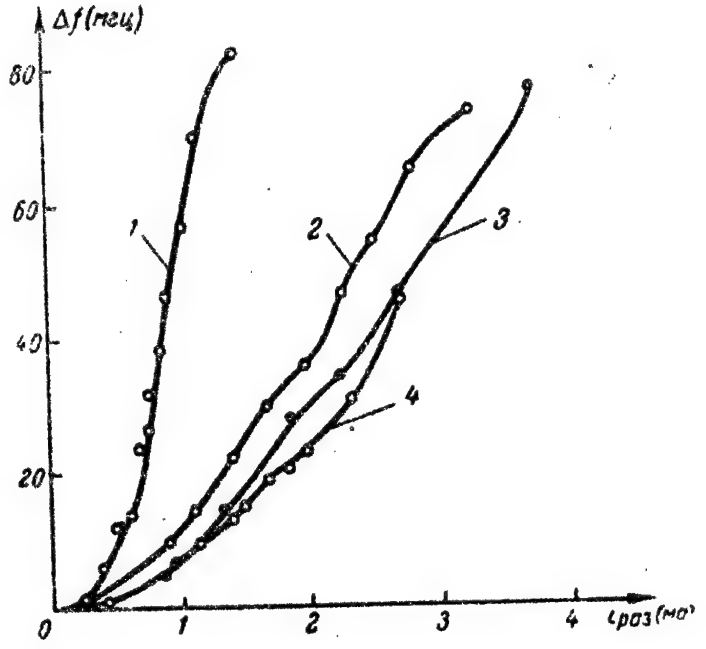

Fig. 7: Resonator retuning as a function of the discharge current in the 10 cm range (neon-filled):

1) p = 3.7 mm Hg, Q_o = 640, Q = 230;
2) p = 2.1 mm Hg, Q_o = 630, Q = 310;
3) p = 1.7 mm Hg, Q_o = 420, Q = 230;
4) p = 5.8 mm Hg, Q_o = 560, Q = 290.

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Moscow State University

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AVERAGED EQUATIONS OF MOTION OF CHARGED PARTICLES IN SLIGHTLY INHOMOGENEOUS STATIC AND HIGH-FREQUENCY FIELDS

Pages 438-442

By M.A. Miller

Averaged equations of motion of charged particles are derived for systems in which besides high-frequency fields there are slightly inhonogeneous electrostatic and magnetostatic fields.

In article [1] an averaged equation of motion was obtained for a charged particle in a slightly inhomogeneous high-frequency field and in a homogeneous static field. It is also of interest to investigate systems in which not only the high-frequency fields but also the static fields are inhomogeneous. Averaged equations of motion, correspondingly generalized, are derived below.

1. Inhomogeneous Electrostatic field.

Let us assume that besides a slightly inhomogeneous high-frequency field

$$\begin{aligned} \boldsymbol{E}_1(\boldsymbol{r}, t) &= \boldsymbol{E}_1(\boldsymbol{r})\, e^{i\omega t} = \boldsymbol{E}_1(\boldsymbol{R})\, e^{i\omega t} + (\boldsymbol{\rho}\nabla)\, \boldsymbol{E}_1(\boldsymbol{R})\, e^{i\omega t}; \\ \boldsymbol{H}_1(\boldsymbol{r}, t) &= \boldsymbol{H}_1(\boldsymbol{r})\, e^{i\omega t} = \boldsymbol{H}_1(\boldsymbol{R})\, e^{i\omega t} + (\boldsymbol{\rho}\nabla)\, \boldsymbol{H}_1(\boldsymbol{R})\, e \end{aligned} \qquad (1)$$

the system also contains a slightly inhomogeneous electrostatic field

$$\boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{E}_0(\boldsymbol{R}) + (\boldsymbol{\rho}\nabla)\, \boldsymbol{E}_0(\boldsymbol{R}), \qquad (2)$$

so that $\underline{E}(\underline{r},t) = \underline{E}_0(\underline{r}) + \underline{E}(\underline{r},t)$ while $E_1 \gg H_1$ or $E_1 \sim H_1$. If the characteristic distance L_0, at which field (2) changes materially, not only considerably exceeds the amplitude of a rapidly oscillating particle $\rho = -(\eta/\omega^2)\underline{E}_1$ but also satisfies the condition

$$\frac{\rho}{L_1} \sim \frac{\rho}{L_0}\frac{E_0}{E_1} \ll 1, \tag{3}$$

where L_1 is the characteristic distance of amplitude change of the high-frequency field, then the contribution of the second term of (2) to the averaged equation of motion can be neglected and the equation can be written in the form [1]:

$$\ddot{R} = -\eta\nabla\varphi(R) - \nabla\Phi(R), \tag{4}$$

where φ and Φ are the electrostatic and the high-frequency potentials respectively:

$$E_0 = -\nabla\varphi;$$
$$\Phi = (\eta/2\omega)^2 |E_1|^2. \tag{5}$$

If condition (3) is not observed, then the inhomogeneous static field must already be taken into account in the equation of rapidly oscillating motion

$$\ddot{\rho} - \eta(\rho\nabla)E_0(R) = \eta E_1(R)e^{i\omega t}, \tag{6}$$

Seeking a solution of this equation and averaging the initial equation (2,3) in article [1]

$$\ddot{R} + \ddot{\rho} = \eta E(R,t) + \eta(\rho\nabla)E(R,t) + \eta/c\,|\dot{R}H(R,t)| + \\ + \eta/c\,|\dot{\rho}H(R,t)| + \eta/c\,|\dot{R}(\rho\nabla)H(R,t)| + \eta/c\,|\dot{\rho}(\rho\nabla)H(R,t)| \tag{7}$$

it is possible to obtain a generalization of the averaged equation (4). It must be borne in mind, however, that in the expansion of (1) only terms of the order $\rho E_1/L_1$ are retained. Therefore in the case $(\rho/L_0)(E_0/E) \sim 1$ and $L_0 \sim L_1$ the

expansion of (2) must be extended to include the terms $(\rho/L_0)^2 E_0$ which are of the same order of magnitude as the terms $(\rho/L_1) E_1$ taken into account in equation (7). A correct treatment within the limits of the accepted approximation is only possible given the additional condition $L_1 \cdot \rho \cdot E_0/L_0^2 E_1 \ll 1$. However, if we give up the assumption that $\rho E_0/L_0 E_1 \sim 1$ and consider that $\rho E_0/L_0 E_1 \ll 1$, then in a first approximation the solution of equation (6) for this small parameter can be written in the form:

$$\rho = \frac{\eta}{\omega^2} E_1(R) e^{i\omega t} + \frac{\eta_1^2}{\omega^4} (E_1 \nabla) E_0 e^{i\omega t}. \qquad (8)$$

By substituting (8) in (7) and averaging over the period $2\pi/\omega$ we get*:

$$\ddot{R} = -\eta_1 \nabla \varphi(R) - \nabla \Phi + \frac{1}{2} \frac{\eta^2}{\omega^4} \operatorname{Re} \{ [(E_1 \nabla) E_0 \nabla] E_1^* + [(E_1 \nabla) E_0 \operatorname{rot} E_1^*] \}. \qquad (9)$$

In the general case the expression in the right-hand member of (9) cannot be represented as a potential vector. It is true that there are some important exceptions. For instance, if the high-frequency field has the same structure as the static one, i.e. $\underline{E}_1(\underline{r}) = \text{const}\, \underline{E}_0(\underline{r})$, then the last term in (9) is written as $(1/8)(\eta^3/\omega^4)\,\text{const}|^2 \nabla(E_0 \nabla |E_0|^2)$ and, consequently,

$$\ddot{R} = -\nabla \Phi_1,$$

where

$$\Phi_1 = \frac{\eta^2}{4\omega^2} |E_1|^2 + \eta_1 \left[\varphi + \frac{\eta_1^2}{8\omega^4} |\text{const}|^2 \nabla (\nabla \varphi \nabla |\nabla \varphi|^2) \right].$$

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* It is assumed that the frequency ω and its subharmonics are remote from the natural frequencies of the homogeneous equation (6).

If necessary, equation (9) may be made more accurate in a similar way by taking into account terms of a higher order with respect to $(\rho/L_0)(E_0/E_1)$.

2. Inhomogeneous Magnetostatic Field

Let us examine the case of simultaneous action of high-frequency and magnetostatic slightly inhomogenous fields. Let us represent the latter in a form analogous to (2):

$$\boldsymbol{H}_0(\boldsymbol{r}) = \boldsymbol{H}_0(\boldsymbol{R}) + (\boldsymbol{\rho}\nabla)\boldsymbol{H}(\boldsymbol{R}). \tag{10}$$

Assuming $|\rho| \ll L_0$, we shall, however, retain terms of the order $(\rho/c)(\rho H_0/L_0 H_1)$ in the average equation, i.e. consider that $H_0/L_0H_1 \lesssim 1$ when $L_0 \sim L_1$.

The rapdily oscillating motion described by equation

$$\ddot{\boldsymbol{\rho}} - (\gamma_1/c)|\dot{\boldsymbol{\rho}}\boldsymbol{H}_0| = \gamma_1 \boldsymbol{E}_1(\boldsymbol{R})e^{i\omega t}, \tag{11}$$

must include oscillation with a field frequency $\rho_\omega(t) = \rho_\omega e^{i\omega t}$ and with a cyclotron frequency $\rho_{\omega_H}(t) = \rho_{\omega_H} e^{i\omega_H t}$ ($\omega_H = \gamma_1 H_0/c$), т. е. i.e.

$$\boldsymbol{r}(t) = \boldsymbol{R}(t) + \boldsymbol{\rho}_\omega(t) + \boldsymbol{\rho}_{\omega_H}(t). \tag{12}$$

Depending on the relationship between the frequency ω and the frequency ω_H it is necessary either first to average equation (7) over the period $2\pi/\omega$ ($\omega \gg |\omega_H|$; the motion $\rho_{\omega_H}(t)$ is then combined with the slow motion $R(t)$) or when $|\omega_H| \gg \omega$ first to average over the period $2\pi/|\omega_H|$ taking into account the slow motion $\rho_\omega(t)$, and then to average afresh over the period $2\pi/\omega$ and over the period of difference frequencies*.

* When ω approaches ω_H or ω_H/n, the averaged equations, thus obtained, become untrue, because in the time of averaging which increases without limit the particles may move through distances exceeding L_1.

However, because the equation for $\underline{R}(t)$, before being averaged over $2\pi/|\omega_H|$, itself has a solution corresponding to rotation of the particle with cyclotron frequency, it is possible purely formally to combine (t) with the unaveraged function R(t) in the second case also:

$$R_{\omega_H}(t) = R(t) + \rho_{\omega_H}(t)$$

and to consider only a forced solution of (11), writing it in projections onto the orthogonal co-ordinate directions τ_1, $\tau_2, \tau_3 (\rho_3 \parallel \underline{H}_0)$ in the following form:

$$\rho_\omega = a_1\tau_1 + a_2\tau_2 + a_3\tau_3;$$

$$a_1 = -\frac{\eta_1}{\omega\sqrt{2}}\left(\frac{E_\perp^{(+)}}{\omega+\omega_H} + \frac{E_\perp^{(-)}}{\omega-\omega_H}\right); \qquad (13)$$

$$a_2 = \frac{i\eta}{\omega\sqrt{2}}\left(\frac{E_\perp^{(+)}}{\omega+\omega_H} - \frac{E_\perp^{(-)}}{\omega-\omega_H}\right);$$

$$a_3 = -\frac{\eta}{\omega^2}E_1.$$

Here the subscript $\parallel$ denotes the component of $\underline{E}_1$, along the direction τ_3, and the subscript $\perp$ denotes the components at right angles to $\underline{H}_0$; the latter in their turn are represented as the sum of vectors with right-handed ($\underline{E}_\perp^{(+)}$) and left-handed ($\underline{E}_\perp^{(-)}$) circular polarization:

$$E_\perp^{(+)} = \frac{1}{\sqrt{2}}E_-^{(+)}(\tau_1 - i\tau_2);\ E_\perp^{(-)} = \frac{1}{\sqrt{2}}E_\perp^{(-)}(\tau_1 + i\tau_2). \qquad (14)$$

By substituting (13) in (9) and averaging over the period $2\pi/\omega$ we get:

$$\overline{(\ddot{R}_{\omega_H} - \omega_H[\dot{R}_{\omega_H}\tau_3])^{2\pi/\omega}} = -\eta_1/4\omega^2\nabla|E_1|^2 - \qquad (15)$$

$$-\frac{\eta^2}{4\omega}\left\{\frac{\nabla|E_\perp^{(+)}|^2}{\omega+\omega_H} + \frac{\nabla|E_\perp^{(-)}|^2}{\omega-\omega_H}\right\} - \frac{\omega}{2}\operatorname{Im}\left[\rho_\omega\left(\rho_\omega^*\nabla\right)\omega_H\tau_3\right].$$

The right-hand member of (15), as in the case of an inhomogeneous electrostatic field, is not in general a potential vector. The following transformations are intended to separate its rotational and potential terms in a clear form.

Let us consider the expression

$$F = \operatorname{Im} |\rho_\omega (\rho_\omega^* \nabla) \omega_H \tau_3|,$$

representing it as a sum of three vectors

$$F = F^{(1)} + F^{(2)} + F^{(3)};$$

$$\begin{aligned} F^{(1)} &= \operatorname{Im} a_2 a_3^* \{ |\tau_2 (\tau_3 \nabla) \omega_H \tau_3| - |\tau_3 (\tau_2 \nabla) \omega_H \tau_3| \}; \\ F^{(2)} &= \operatorname{Im} a_1 a_3^* \{ |\tau_1 (\tau_3 \nabla) \omega_H \tau_3| - |\tau_3 (\tau_1 \nabla) \omega_H \tau_3| \}; \\ F^{(3)} &= \operatorname{Im} a_1 a_2^* \{ |\tau_1 (\tau_2 \nabla) \omega_H \tau_2| - |\tau_2 (\tau_1 \nabla) \omega_H \tau_3| \}. \end{aligned} \tag{16}$$

Starting from the equations of a magnetostatic field*,

$$\operatorname{rot} \omega_H \tau_3 = 0; \quad \operatorname{div} \omega_H \tau_3 = 0,$$

and taking into consideration that the vectors τ_1, τ_2, τ_3 are orthogonal

$$|\tau_1 \tau_2| = \tau_3; \quad |\tau_2 \tau_3| = \tau_1; \quad |\tau_3 \tau_1| = \tau_2,$$

one can derive the relationships:

$$\begin{aligned} &\operatorname{rot} \tau_3 = -(1/\omega_H) |\nabla \omega_H \tau_3|; \quad \operatorname{div} \tau_3 = -(1/\omega_H)(\tau_3 \nabla \omega_H); \\ &(\tau_3 \nabla) \tau_3 = (1/\omega_H) \nabla \omega_H; \quad \tau_3 (\tau_2 \nabla) \tau_3 = -\tau_1 \operatorname{rot} \tau_2; \\ &\tau_1 (\tau_1 \nabla) \tau_3 = \tau_3 \operatorname{rot} \tau_1; \\ &\tau_1 (\tau_2 \nabla) \tau_3 = \tau_2 (\tau_2 \nabla) \tau_3 = \tau_2 (\tau_1 \nabla) \tau_3 = \tau_3 (\tau_1 \nabla) \tau_3. \end{aligned} \tag{17}$$

By projecting (16) onto the directions τ_1, τ_2, τ_3, after a number of transformations, using (17), we get:

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* It is assumed that no sources are contained in the region under examination.

$$F^{(1)} + F^{(2)} = -\operatorname{Im}\{\tau_1 a_2 a_3^* (\tau_3 \operatorname{rot} \tau_1) + \\ + \tau_2 a_1 a_3^* \omega_H (\tau_1 \operatorname{rot} \tau_2) + \tau_3 (a_2 a_3^* \tau_1 \nabla \omega_H - a_1 a_3^* \tau_2 \nabla \omega_H)\}; \\ F^{(3)} = -\operatorname{Im} a_1 a_2^* \nabla \omega_H. \tag{18}$$

It is not difficult to satisfy oneself that equations (18) are equivalent to the following relationship:

$$F = -\frac{1}{2} \operatorname{Im} (|\rho_\omega \rho_\omega^*| \nabla) \omega_H \tau_3. \tag{19}$$

Thanks to such notation the physical interpretation of the last term in equation (15) becomes obvious. It is due to the force acting from the inhomogeneous magnetostatic field on the magnetic dipole $\overline{p^m}$, which is constant with respect to time and which appears as the result of averaging the oscillating motion of the particle:

$$\overline{p^m} = \frac{e}{2c} \overline{|\rho_\omega \dot{\rho}_\omega|} = \frac{e\omega}{4c} \operatorname{Im} |\rho_\omega \rho_\omega^*|. \tag{20}$$

In fact

$$-\frac{\omega}{2} \operatorname{Im} |\rho_\omega (\rho_\omega \nabla) \omega_H \tau_3| = \frac{\omega}{4} \operatorname{Im} (|\rho_\omega \rho_\omega^*| \nabla) \omega_H \tau_3 = \\ = \frac{c}{e} (\overline{p^m} \nabla) \omega_H \tau_3 = \frac{1}{m} (\overline{p^m} \nabla) H_0. \tag{21}$$

By representing the dipole moment (20) as the sum of the longitudinal ($\overline{p_{\parallel}^m}$) and transverse ($\overline{p_{\perp}^m}$) components and taking (21) into account, we can combine the potential terms on the right-hand side of equation (15), writing it in the following form:

$$\overline{R_{\omega_H} - \omega_H |R_{\omega_H} \tau_3|^{\frac{2\pi}{\omega_H}}} = -\nabla \Phi + \frac{1}{m} (\overline{p_{\perp}^m} \nabla) H_0, \tag{22}$$

where

$$\Phi = \left(\frac{\gamma_1}{2\omega}\right)^2 \left\{ |E_\parallel|^2 + \frac{\omega}{\omega+\omega_H} |E_\perp^{(+)}|^2 + \frac{\omega}{\omega-\omega_H} |E_\perp^{(-)}|^2 \right\}; \tag{23}$$

$$\overline{p_\perp^m} = \frac{e\omega}{2c} \operatorname{Im} \rho_{\tau_3}(\rho^*_{\tau_1}\tau_2 - \rho^*_{\tau_2}\tau_1). \tag{24}$$

Formally the high-frequency potential (23) does not differ from the potential for systems with a homogeneous magnetic field, but the cyclotron frequency ω_H entering into it is a function of the coordinates. When ω_H = const the last term of equation (22) vanishes*, and its right-hand member becomes potential [1]. It is also potential when $\underline{E}_1 \perp \underline{H}_0$, because in this case, according to (13), $\rho_{\tau_3} = a\,\tau_3 = 0$ i.e. the particle rotates only in planes perpendicular to the lines of H_0.

It remains to make the significance of averaging the left-hand member of (22) more precise. As already mentioned, when $\omega \gg |\omega_H|$, the function $\rho_{\omega_H}(t)$ describes slow (on the scale of $2\pi/\omega$) motion, therefore if averaging over the period $2\pi/|\omega_H|$ is not of special interest, equation (22) can be used as an ordinary averaged equation without any reservations. When ω approaches $|\omega_H|$ and also when $\omega \ll |\omega_H|$, it is necessary

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* The presence of $\overline{p_\perp}$ causes a precession of the magnetic dipole in a homogeneous magnetic field but in the approximation in question it does not lead to a displacement of the center of oscillation; therefore the precession is averaged and drops out of the equation for $\underline{R}(t)$. When motion is studied in a quasi-stationary magnetic field $H_1 \gg E_1$ and in a homogeneous constant magnetic field it is imperative to take precession into account (see, for example, article [2]).

in principle, to average the left-side of (22) over the period $2\pi/|\omega_H|$ and over the periods of the difference frequencies; we know that this leads to the so-called equation of drift approximation [3]. Therefore, when $\omega \ll |\omega_H|$, equation (22) actually reduces to the following:

$$\ddot{R} - \omega_H [\dot{R}\, \tau_3] = -\nabla \Phi + \frac{1}{m} (\overline{p_1^m} \nabla) H_0 + \frac{1}{m} (p_{0_\parallel}^m \nabla) H_0, \qquad (25)$$

$$p_{0_H}^m = \frac{e}{4c} \operatorname{Re} [\rho_{\omega_H} \dot{\rho}^*_{\omega_H}] = -\frac{\omega_H}{4} |\rho_{\omega_H}|^2 \tau_3;$$

where $$\rho_{\omega_H} = \rho_{\omega_H} / \sqrt{2}(\tau_1 + i\tau_2)\, e^{i\omega_H t}.$$

Here $\underline{R}(t)$, in contrast to $\underline{R}_{\omega_H}(t)$, is a slow (on the scale of $2\pi/\omega_H$) function of time.

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EXCITATION OF A TRANSMISSION LINE WITH IRRECTILINEAR ELECTRON BEAMS*

Pages 443-449

By A.V. Gaponov

The author examines the problem of the excitation of a transmission line with an irrectilinear electron beam, in which the particles oscillate not only longitudinally but also transversely. He explains under what conditions the electron beam excites monochromatic waves in the system. An expression is obtained for the amplitudes of normal waves in terms of the coordinates of electrons in the beam.

When a high-frequency field in a transmission line (waveguide) interacts with an electron flux, not only longitudinal but also transverse oscillations are in general excited. It is not difficult to see that electron beam oscillations in directions perpendicular to the static trajectories of the electrons lead to the density of the convection current in a fixed frame of reference not being everywhere a sinusoidal function of time, even for interaction with a monochromatic wave, however weak, when each individual electron in the beam oscillates harmonically. This circumstance, i.e. the nonsinusoidal relation of current density to time, makes it difficult to find a self-consistent solution of the problem of interaction between an electron beam and a monochromatic wave. On the other hand, it is more or less evident that under the action of a weak

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monochromatic high-frequency field there is just that distribution of current density (nonsinusoidally time-dependent) set un in the beam, in a linear approximation, which excites a monochromatic wave in the line. It therefore seems desirable to find out under what conditions the electromagnetic field, which is excited in a transmission line by an electron beam with a given law of velocity modulation of particles, can be represented as a superposition of monochromatic normal waves (of identical frequency), and also to find expressions for the amplitude of these waves in terms of the characteristics of an electron beam which can be most easily determined (for example in terms of the coordinates and velocity of the particles).

The excitation of a waveguide by an arbitrarily time-dependent current is fully described by the "waveguide equations" in [1]. However, as these equations are most cumbersome, we shall first study this problem in a quasi-stationary approximation (for transmission lines with high delay) and shall then extend the results obtained to any waveguide system.

1. Excitation of a Cylindrical Transmission Line (quasi-stationary approximation)

Let us consider a transmission line penetrated by an electron beam (an example of this kind of line is shown in Fig. 1) and let us assume that the line's transverse dimensions d and the length of the fundamental wave in it $\lambda_в$ are so small that the electric field can be considered potential:

$$E = -\nabla\varphi; \qquad \varphi = V(z,t)\Psi(xyz) + \varphi_p. \tag{1}$$

Here φ_p is the potential of the Coulomb field of the beam's space charge, distributed with a density $\rho(xyz,t)$, V(z,t) is the voltage in the line $\psi(xyz)$ is the static potential distribution function, periodic with respect to z (with period D).

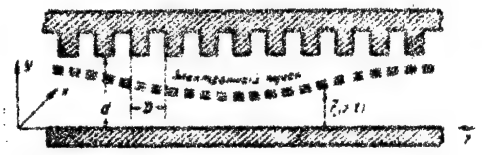

Fig. 1.

Let us further assume that $d, D \ll \lambda_b$ (assuming that Ψ only depends on the transverse coordinates: $\Psi(x,y)$). If we set up telegraph equations for the line current $J(z,t)$ and for the voltage $V(z,t)$ and if with the aid of the reciprocity theorem we take into account the charge induced in the conductors by the beam's space charge, we get

$$\frac{\partial V}{\partial z} + L_1 \frac{\partial J}{\partial t} = 0; \qquad \frac{\partial J}{\partial z} + C_1 \frac{\partial V}{\partial t} = M(z,t), \tag{2}$$

where L_1 and C_1 are linear parameters self-inductance and capacitance of the line

$$M(z,t) = \frac{\partial}{\partial t} \int_{S_\perp} \rho(xyz,t)\Psi(x,y)dS \tag{3}$$

(the integration is with respect to the cross section of the line). If we eliminate the current J from (2), then we get an inhomogeneous wave equation for the voltage $V(z,t)$:

$$\frac{\partial^2 V}{\partial z^2} - L_1 C_1 \frac{\partial^2 V}{\partial t^2} = - L_1 \frac{\partial}{\partial t} M(z,t). \tag{4}$$

It follows from equations (3) and (4) that the electron beam excites a monochromatic wave in the system, if $M(z,t)$ is a sinusoidal function of time. Because in this case the charge density $\rho(xyz,t)$ may have a complicated relation to

time, the Coulomb field of the space charge E_p, which is determined by the equations

$$E_p = -\nabla\varphi; \qquad \nabla\varphi = -\frac{1}{\varepsilon}\rho(x,y,z,t), \tag{5}$$

will not in general be a harmonic function of time in the neighborhood of the beam. This makes a consistent calculation of the space charge field more difficult. It should be noted, however, that in the majority of cases of interest in practice the electron beam appears grouped in such a way that resonance (coherent) excitation of a wave of definite type is brought about in the transmission line. Also the amplitude of the wave appears to be large enough to neglect the space charge field in the first approximation*.

Let us examine the expression for $M(z,t)$ in greater detail. By bringing the differentiation with respect to time under the integral sign and using the continuity equation it is not difficult to express $M(z,t)$ in terms of the longitudinal and transverse components of the current density j in the beam, i.e. $j_\parallel$ and $j_\perp$:

$$M = -\int_{S_\perp} \Psi \operatorname{div} \boldsymbol{j}\, dS = -\frac{\partial}{\partial z}\int_{S_\perp} j_\parallel \Psi\, dS + \int_{S_\perp} \boldsymbol{j}_\perp \nabla\Psi\, dS. \tag{6}$$

If the beam is sufficiently thin** and the function

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* For an approximate calculation of the space charge field we can confine ourselves to the first time harmonic of the solution of equation (5).

** This assumption is not one of principle but simplifies the arrangement.

of potential distribution function $\Psi(x,y)$ and its derivatives change little within the cross section of the beam, it is possible to carry out integration with respect to the cross section in (3) or (6). We shall characterize a thin bean by the equation $r_\perp = r_\perp(z,t)$, which determines its shape (see fig. 1), by the longitudinal velocity of the particles $v_z = v_z(z,t)$ and by the longitudinal (with respect to axis z) charge $\rho_{nor} = \rho_{nor}(z,t)$. Assuming that there are no intersecting trajectories in the beam and no overtaking of some electrons by others, we shall consider $r_\perp$, v_z and ρ_{nor} to be single-valued functions of their arguments. For M(z,t) we shall then get:

$$M(z,t) = -\Psi(r_\perp)\frac{\partial I_\parallel}{\partial z} + \rho_{nor}\nabla\Psi(r_\perp)\frac{\partial r_\perp}{\partial t}, \tag{7}$$

where $I_\parallel = \rho_{nor} v_z$ is the longitudinal current across the section z = const, related to the velocity v_z by the equation

$$\frac{\partial I_\parallel}{\partial z}v_z^2 + \frac{\partial I_\parallel}{\partial t}v_z = I_\parallel\frac{\partial v_z}{\partial t}. \tag{8}$$

Let us assume that the stationary electron beam ($I_\parallel = I_0 = \text{const}$, $r_\perp = r_\perp^{(o)}z$, $v_z = v_z^{(o)}(z)$) is modulated by small oscillations, i.e. that

$$r_\perp = r_\perp^{(0)}(z) + r_\perp^{(1)}(z,t); \qquad v_z = v_z^{(0)}(z) + v_z^{(1)}(z,t),$$

where

$$|v_z^{(1)}| \ll v_z^{(0)}, \qquad |r_\perp^{(1)}| \ll d, \qquad d \sim |\Psi|/|\nabla\Psi|$$

is the scale of inhomogeneity of the high-frequency field. Then it is not difficult to see from (8) that the variable component of the longitudinal current is also small:

$$I_\parallel = I_0 + I_\parallel^{(1)}(z,t), \qquad |I_\parallel^{(1)}| \ll |I_0|.$$

By linearizing expressions (7), (8) it is not difficult to satisfy oneself that the unperturbed (stationary) electron beam, as should be expected, does not excite a wave in the transmission line ($M^{(o)} \equiv 0$). With a sinusoidal perturbation, i.e. when $r_\perp^{(1)} = r_{\perp a}^{(1)}(z)\, e^{i\omega t}$, $V_z^{(1)} = V_a^{(1)}(z)\, e^{i\omega t}$, the value $M(z,t)$ is also a harmonic function of time: $M = M_a(z)e^{i\omega t}$. A monochromatic wave $V(z,t) = V_a(z)e^{i\omega t}$ is then excited in the line and its amplitude is determined according to (4) by the equations:

$$\frac{d^2V_a}{dz^2} + h_0^2 V_a = -i\omega L_1 M_a(z); \tag{9}$$

$$M_a(z) = -\Psi(r_\perp^{(0)})\frac{\partial I_{\parallel a}^{(1)}}{\partial z} + i\omega\frac{I_0}{v_z^{(0)}}\Delta\Psi(r_\perp^{(0)})\, r_{\perp a}^{(1)}; \tag{10}$$

$$(v_z^{(0)})^2\frac{\partial I_{\parallel a}^{(1)}}{\partial z} + i\omega v_z^{(0)} I_{\parallel a}^{(1)} = i\omega I_0 v_{za}^{(1)}, \tag{11}$$

where $h_o^2 = \omega^2 L_1 C_1$ is the wave propagation constant in the absence of an electron beam.

2. Excitation of a Cylindrical Transmission Line (The general case)

By using "waveguide equations" (see for instance [1]) it is not difficult to extend the results obtained above in a quasi-stationary approximation to any cylindrical transmission line. In particular, it can be shown that in the presence of small sinusoidal (with respect to time) perturbations in the beam the S type wave, excited in the line by such a beam, will be monochromatic, provided the amplitude of the oscillations of the particles in the beam is much smaller than the characteristic dimension of inhomogeneity of the field of this wave. For the amplitudes of the monochromatic waves we shall get equations

analogous to (4):

$$\frac{d^2V_s}{dz^2} + h_{0s}^2 V_s = -\frac{2ih_{0s}}{N_s} M_s(z), \qquad (12)$$

where N_s is the wave normal [2] ($N_s > 0$) and the $M(z,t) = M_s(z)e^{i\omega t}$ for TM and TE waves are determined by the expressions:

$$M^{TM} e^{i\omega t} = \left[\frac{x_s^2}{h_{0s}^2} \frac{\partial}{\partial z} \int_{S_\perp} j_h \Psi_s^e dS + \int_{S_\perp} j_\perp \nabla \Psi_s^e dS \right]_\omega ;$$
$$M_s^{TE} e^{i\omega t} = \left[-\int_{S_\perp} j_\perp \left[z_0 \nabla \Psi_s^m \right] dS \right]_\omega . \qquad (13)$$

Here ψ_s^e and ψ_s^m are two-dimensional membrane functions (proportional to the longitudinal components of the electric and magnetic fields in normal waves), z_0 is the unit vector along the Z-axis, $x_s^2 = \omega^2 \varepsilon\mu - h_{os}^2$ are the transverse wave numbers, ε, μ are the dielectric constant and the magnetic permeability. The symbol $[\ ]_\omega$ denotes the Fourier component of the frequency ω of corresponding value. The fields in the transmission line are written in terms of $V_s(z)$, ψ_s^e, and ψ_s^m; for example for TM waves

$$E_z^s = \frac{\partial V_s}{\partial z} \frac{x_s^2}{h_{0s}^2} \Psi_s^e ; \quad E_{tr}^s = -V_s \nabla \Psi_s^e ; \quad B_{tr}^s = -\frac{\partial V_s}{\partial z} \frac{i\omega\varepsilon\mu}{h_{0s}^2} [z_0 \nabla \Psi_s^e], \qquad (13a)$$

and for TE waves

$$H_z^s = V_s \frac{x_s^2}{i\omega\mu} \Psi_s^m ; \quad H_{tr}^s = \frac{\partial V_s}{\partial z} \frac{1}{i\omega\mu} \nabla \Psi_s^m ; \quad E_{tr}^s = V_s [z_0 \nabla \Psi_s^m] . \qquad (13б)$$

In the linear approximation in question, when the amplitude of the oscillations of particles in the electron beam is assumed to be small and the oscillations themselves to be

sinusoidal, the variable component $M(b,t)$ is also a sinusoidal function of time. Note, however, that the limitation to a linear approximation flows, as already mentioned, from the condition $|r^{(1)}_{1a}| \ll |\gamma_s| \, |V^{\mp}_s|$. For waves with a higher order number this condition is evidently not fulfilled and because of this the field in the transmission line is not strictly monochromatic*.

It is known that the solutions of equation (10), (12) may be written in the form of a superposition of two opposing waves: $V_s(z) = V_s^+(z)e^{-ih_{os}z} + V_s^-(z)e^{ih_{os}z}$. If the transmission line is infinite (or matched in both directions) then for a high-frequency field in the line it is not difficult to get:

$$E = \sum_s (V_s^+ E_s^+ + V_s^- E_s); \qquad B = \sum_s (V_s^+ B_s^+ + V_s^- B_s^-). \qquad (14)$$

where

$$V_s^{\pm} = \mp \frac{1}{N_s} \int_{z_{1,2}}^{z} \left[\int_{S_1} J(xyz,t)\, E_s^{\mp}(xyz)\, dS \right] dz , \qquad (15)$$

and $E_s^{\pm} = e_s^{\pm} e^{\mp ih_{os}z}$, $B_s^{\mp} = b_s^{\pm} e^{\mp ih_{os}z}$ are the electric field strength and the magnetic flux density in the direct ($e^{-ih_{os}z}$) and opposing ($e^{+ih_{os}z}$) normal waves; $Z_{1,2}$ are the coordinates of the beginning and end of the electron beam.

Note that these expressions formally coincide with known formulas [3], which describe the excitation of a waveguide by a sinusoidal current $j = j(xyz)\, e^{i\omega t}$; they are useful

* In a quasi-stationary approximation the sum of fields of waves with high order numbers corresponds to the Coulomb field of the space charge in the neighborhood of the beam.

for any waves (not only TE and TM) in transmission lines.

Substituting $j = \rho v$ in expression (15) for the amplitudes of the normal waves and assuming that the electron beam is sufficiently thin, after integrating with respect to cross section of the waveguide:

$$V_s^{\pm} = \mp \frac{1}{N_s} \int_{z_{1,2}}^{z} \left\{ \rho_{nor} \, v \, E_s^{\mp} [r_{\perp}(z, t), z] \right\}_{\omega} dz, \qquad (16)$$

where $E_s^{+} [r_{\perp}(z,t), z]$ is the value of the field of the normal wave in that point of the cross section through which the electron beam is passing at the given moment.

3. Expressions for the Amplitudes of the Normal Waves in Terms of the Coordinates and Velocity of electrons in the Beam

When we derived equations (12), (13) and their equivalent formulas (14), (16), describing the excitation of a transmission line by an electron beam, we assumed that the shape of the beam $r_{\perp}(z,t)$, the charge density $\rho_{nor}(z,t)$ and the longitudinal velocity of the electrons $v_z(z,t)$ were given in Euler's variables. Nevertheless, in a number of cases it turns out to be more convenient to describe the electron beam in Lagrangian variables t_B, t (t_B being the time the electron enters the system) or in the variables τ, t where τ is the time during which the electron remains in the system (time of transit). In these variables formulas (16) for the normal wave amplitudes take on a specially simple form.

For the transition to variables τ, t we shall use the relation $z = z(\tau, t)$ (or $\tau = \tau(z,t)$), which can be found from the equations of motion of an electron which in what follows we assume known. If the beam is not modulated at the input, then,

as known, $\rho_{nor} = |I_0|\left(\frac{\partial z}{\partial \tau}\right)_t = -|I_e|\left(\frac{\partial z}{\partial \tau}\right)_t$ (see for example [3]) and transition to the new variables in (16) gives:

$$V_s^{\pm} = \pm \frac{|I_0|}{N_s}\left\{\int_{\tau(z_0,t)}^{\tau(z,t)} v(\tau,t)\, E_s^{\mp}\,|r_\perp(\tau,t), z(\tau,t)|\, d\tau\right\}_{\omega}. \qquad (17)$$

Thus, it appears possible to express the amplitudes of the normal waves solely in terms of the total beam current I_0 and the electron velocity $v(\tau,t)$. The generalization of (17) for beams of finite thickness, taking into account the limitations of the first section, presents no difficulty.

In a slightly perturbed electron beam the coordinates r and the velocities $v \equiv \dot{r}$ of the electrons are close to the coordinates $r^{(0)}$ and velocities $v^{(0)}$ in the stationary beam:

$$\begin{aligned} &r = r^{(0)}(\tau) + r^{(1)}(\tau,t), && |r^{(1)}| \ll |E|/|\nabla E|; \\ &\dot{r} \equiv v = v^{(0)}(\tau) + v^{(1)}(\tau,t), && |v^{(1)}| \ll |v^{(0)}|; \\ &\tau = \tau^{(0)}(z) + \tau^{(1)}(z,t), && |\tau^{(1)}| \ll |\tau^{(0)}|. \end{aligned} \qquad (18)$$

By using the smallness of the perturbations it is possible to separate the variable component on the right-hand side of (17) in a clear form. Let us consider the perturbations sinusoidal ($r^{(1)} = r_\tau^{(1)}(\tau)e^{i\omega t}$, $v^{(1)} = v_\tau^{(1)}(\tau)e^{i\omega t}$ and expand the field under the integral in (15) in series in the point $r = r^{(0)}$. Confining ourselves to the first terms of the expansion, after several transformations we get:

$$V_s^{\pm} = \pm \frac{|I_0|}{N_s}\left\{i\omega \int_{\tau^{(0)}(z_{1,2})}^{\tau^{(0)}(z)} r_\tau^{(1)}\,(E_{s\perp}^{\pm})^* d\tau + E_s^{\mp} v^{(0)} \tau^{(1)} \Big|_{\tau^{(0)}(z_{1,2})}^{\tau^{(0)}(z)} + E_s^{\mp} r^{(1)} \Big|_{\tau^{(0)}(z_{1,2})}^{\tau^{(0)}(z)}\right\}. \qquad (19)$$

Here * means a complex conjugate quantity $\left|_{\tau^{(0)}(z_{1,2})}^{\tau^{(0)}(z)}\right.$ is a substitution within the corresponding limits and $E_{s\bar{a}}^{\pm}$ stands for the variable field acting on the electron in the tracking frame of reference (in a stationary approximation).

$$E_{s1}^{*} = E_{s}^{\pm} + |\tau^{(0)} B_{s}^{\pm}|. \qquad (20)$$

Formula (19) is very convenient for calculating the field excited in a transmission line by an electron beam, because it gives an expression for normal wave amplitudes directly in terms of the coordinates of an individual electron $\underline{r}^{(1)}(t)$ (at points where $v_z^{(o)} \neq 0$, $\tau^{(1)} \approx - z^{(1)}/v_z^{(o)}$); thus there is no need to determine the variable component of the beam current. It is not difficult to see that the second and third addends on the right-hand side of (19) are always small: the amplitude of the high-frequency field corresponding to them is of the order of $V \sim Z_B I_o v^{(1)}/v^{(o)}$ (Z_B is the wave impedance of the line). The first addend is also generally small, but under certain conditions (proper phasing of the electron beam perturbation $\underline{r}^{(1)}$ with one of the natural waves of the transmission line) it can grow infinitely with increase in the length of the system. Therefore, when investigating systems in which such phasing is known to occur (as, for instance, with the interaction of a wave in a transmission line with a space harmonic of the electron beam current, synchronized with it [4], only the first term of the right-hand side of (19) need be taken into account.

In conclusion, note that owing to the equations for the field being linear the formula for the amplitudes of normal waves excited in a line by several thin beams (or by a beam of

finite thickness) can be obtained by summing (integrating) expressions of type (19).

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INTERACTION BETWEEN ELECTRON FLUXES AND ELECTROMAGNETIC WAVES IN WAVEGUIDES

Pages 450-462

By A.V. Gaponov

A solution is obtained for the propagation of an electromagnetic wave in a waveguide containing a thin electron beam that follows a curvilinear path. The dispersion equations are derived for waveguides containing spiral and trochoidal beams, which traverse (respectively) a homogeneous axial magnetic field and crossed electric and magnetic fields. Various modes of interaction between beam and waves are considered, and it is found that microwaves can be generated and amplified in systems not having slow waves.

Introduction

In any study of the interaction between an electron flux and electromagnetic waves it is usual to suppose that the electrons in the beam follow straight paths when the radio-frequency field is absent. That assumption is made in most of the theoretical papers on traveling-wave tubes and the like. But it is known that quite new interaction effects can appear if the beam is velocity-modulated. One such effect is an interaction with the spatial harmonics of the beam current if the electrons have speeds that are a periodic function of the axial coordinate [1-5]. This effect opens up many possibilities, because we can use the interaction between a fast spatial harmonic and a wave of normal speed in a waveguide and in that way avoid the use of retarding systems, whose difficulty of

manufacture constitutes the main obstacle to progress towards shorter wavelengths.

An electric field can be used to produce velocity modulation [1,4,5], for which purpose it has been assumed that the beam is straight and that the axially periodic variation in the velocity is produced by an axial electrostatic field that varies periodically along the axis. The beam can interact with waves of normal speed (amplification and generation can occur), but there remains a periodic structure having a small repeat distance (whose function is to modulate the beam, not to slow down the waves). Again, the beam must be focussed carefully, because the modulating field lies close to the surfaces in the periodic structure.

Another method of modulation not requiring a periodic structure would be better, e.g. one in which a homogeneous transverse or axial magnetic field is used*.

The particles in the beam follow curved paths in most of the systems in which magnetic fields or periodic (or other) electric fields are used to produce the focussing. The problem in its most general form may be formulated as one in which a wave in a waveguide interacts with an electron beam following a curvilinear path determined by steady electric and magnetic fields that vary periodically in space. That very general approach is used here only in deriving the basic equations; the dispersion equation is derived for the case in which the two

* Ludi [2,3] was the first to point out that a cycloidal beam (one guided by crossed constant electric and magnetic fields) can interact strongly with a wave whose phase velocity is greater than the drift velocity of the electrons.

fields are homogeneous.

In an earlier paper [6] ** I have dealt with the way in which an electron beam can excite a waveguide; the formulas given there enable us to express the field excited in the waveguide in terms of the variable (high-frequency) component of the radius vector of a single electron in a thin beam. Therefore we may solve for the interaction between a wave in a waveguide and a thin electron beam by solving the equations for the motion of an electron subject to the fields defined by the expressions given in I.

1. Motion of an Electron in a Fixed High-Frequency Field.

1. I assume that the space-charge density in the beam is so small that the high-frequency field caused by the beam has no effect on the motion of a single electron. Then the path followed by any electron in the beam is the same as that followed by a single electron moving in the same external fields, which are the steady fields $\underline{E}_0(\underline{r})$ and $\underline{B}_0(\underline{r})$ and the high-frequency electromagnetic field varying sinusoidally in time $\underline{E},\underline{B}$. The equation of motion (in the nonrelativistic approximation) is

$$\ddot{\boldsymbol{r}} = -\eta\left[\boldsymbol{E}_0 + \boldsymbol{E} + \left[\dot{\boldsymbol{r}}, \boldsymbol{B}_0 + \boldsymbol{B}\right]\right], \qquad (1)$$

in which $\eta = |e|/m$. The high-frequency field is treated as a perturbing field; the solution $\underline{r}(t)$ is represented as a series

$$\boldsymbol{r} = \boldsymbol{r}^{(0)} + \boldsymbol{r}^{(1)} + \ldots \qquad (|\boldsymbol{r}^{(0)}| \gg |\boldsymbol{r}^{(1)}| \gg \ldots).$$

** In future [6] is denoted by I, which is also used to denote references to formulas in that paper.

We expand the field appearing on the right in (1) as a series for $\underline{r}^{(o)}$ and equate terms of the first order of smallness; we get

$$\ddot{r}^{(0)} + \gamma_1 [\dot{r}^{(0)} B_0] = -\gamma_1 E_0(r^{(0)}); \tag{2a}$$

$$\ddot{r}^{(1)} + \gamma_1 [\dot{r}^{(1)}, B_0(r^{(0)})] + \gamma_1 [\dot{r}^{(0)}, (r^{(1)}\nabla) B_0(r^{(0)})] + \gamma_1 (r^{(1)}\nabla) E_0(r^{(0)}) = -\gamma_1 E_1(r^{(0)}, t), \tag{2b}$$

in which $\underline{E}' = \underline{E} + \left[\underline{r}^{(o)} \underline{B}\right]$ denotes the alternating field acting on the electron (in the coordinate system conjugate in the zeroth approximation.

Equation (2b) is linear (in the first approximation) and has variable coefficients; its right half is time-dependent. Its structure is such that we can distinguish two classes of system that appear to be most simple as regards our studies, namely a) the class in which $\underline{B}_o$ is zero and $\underline{E}_o(\underline{r})$ is inhomogeneous and periodic along z, and b) the class in which $\underline{E}_o$ and $\underline{B}_o$ are homogeneous (one particular case being that in which $\underline{E}_o = 0$).

The first class includes systems in which the beam is focused by means of a periodic electrostatic field; it also includes systems in which any transverse motion of the electrons is excluded by a strong axial magnetic field (the velocity modulation being produced by a periodic electrostatic field). In this case (2b), subject to appropriate initial conditions (2a), becomes an equation having periodic coefficients and a quasiperiodic right half (interaction with an inhomogeneous plane wave). In the simplest case (a straight beam), the vector equation (2b) reduces to a scalar one; the corresponding interaction has been examined by Solnisev and Tager [4, 5].

The second class covers systems in which the beam is

guided by crossed uniform electrostatic and magnetostatic fields or by a uniform axial magnetic field, the spatial modulation being produced only as a result of suitably chosen initial conditions (trochoidal and spiral beams). Then (2b) has constant coefficients and a quasiperiodic right half (when the high-frequency field is periodic in space). We shall consider this class in more detail.

2. Consider a thin electron beam guided by crossed homogeneous fields $\underline{E}_0 = -\underline{y}_0$ and $\underline{B}_0 = \underline{x}_0 B_0$ ($\underline{x}_0$ and $\underline{y}_0$ being unit vectors -- see Fig.1.). We assume that the initial velocity along x is zero. Then zeroth approximation equation

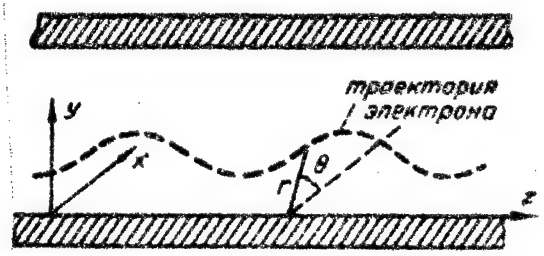

Fig. 1. Thin curvilinear electron beam in a waveguide

$$\ddot{x}^{(0)} = 0; \quad \ddot{y}^{(0)} + \omega_H \dot{z}^{(0)} = \eta E_0; \quad \ddot{z}^{(0)} - \omega_H \dot{y}^{(0)} = 0 \qquad (3)$$

describes a trochoid

$$x^{(0)} = x_0; \quad y^{(0)} = y_0 - a\cos(\omega_H \tau); \quad z^{(0)} = v_0 \tau - a\sin(\omega_H \tau), \qquad (4)$$

in which $v_0 = E_0/B_0$ is the drift velocity, $\omega_H = \eta B_0$ is the gyrofrequency, $\tau = \tau^{(0)} = t - t_i$ is the time of flight (in the zeroth approximation), t_i is the time at which the electron

enters (at the plane $z_1 = 0$), and the constants x_0, y_0, and a are determined by the initial conditions of motion.

The equation for the first approximation in this case takes the form

$$\ddot{r}^{(1)} + \eta [\dot{r}^{(1)} B_0] = -\eta E_1(x^{(0)}, y^{(0)}, z^{(0)}, t). \qquad (5)$$

If the high-frequency field has one frequency only, we may put the righthand side of (5) in the form

$$E_1 = E_1 [x_0, y^{(0)}(\tau), z^{(0)}(\tau)] e^{i\omega t} = E_1(\tau) e^{i\omega t}.$$

It is convenient to seek the solution to (5) in the form of a function of τ and t^*, i.e. in the form $\underline{r}^{(1)} = \underline{r}^{(1)}_{\tau}(\tau) \exp i\omega t$. We substitute for $\underline{r}^{(1)}$ in (5) and differentiate with respect to t; we get an equation defining the amplitude of the perturbation $r^{(1)}_{\tau}$:

$$\frac{d^2 r^{(1)}_\tau}{d\tau^2} + 2i\omega \frac{dr^{(1)}_\tau}{d\tau} - \omega^2 r^{(1)}_\tau + \eta \left[\frac{dr^{(1)}_\tau}{d\tau} + i\omega r^{(1)}_\tau, B_0 \right] = -\eta E_1(\tau). \quad (6)$$

which is a linear equation with constant coefficients; $\underline{E}(xyz)$ and (4) being given, we can solve it in the usual way.

The typical features of the interaction can be revealed by considering the motion of a charged particle in the field of

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* If the beam is modulated by an axial field $E_{0z} = -\partial\varphi/\partial z$ (as in a system falling in the first class), the solution to the zeroth approximation equation (obtained from the law of conservation of energy) takes the form

$$z^{(0)}(z) = \int_0^z dz/v^{(0)}(z), \text{ где } [v^{(0)}(z)]^2 = 2\eta\varphi(z) + v_0^2.$$

Again, the right half of (2b) is put as a function of z and t_1; then $r^{(1)}$ is sought as a function of z and t_1 (or z and t).

a single-frequency wave

$$E = V_0 e(xy)\, e^{-ihz}; \qquad B = V_0 b(xy)\, e^{-ihz}, \tag{7}$$

in which V_0 = const and h is the propagation constant (which in general is complex). The right half of (6) then becomes

$$-\gamma_1 E_x(\tau) = -\gamma_1 V_0 e_x(x_0,\ y_0 - a\cos\omega_H\tau) e^{iah\sin(\omega_H\tau)} e^{-ihv_0\tau} = -\eta V_0 G(\tau)\, e^{-ihv_0\tau}, \tag{8}$$

in which $\underset{\sim}{e}' = \underset{\sim}{e} + [\underset{\sim}{r}^{(0)},\ \underset{\sim}{b}]$ and $G(\tau) = \underset{\sim}{e}' e^{iah\sin(\omega_H\tau)}$ is a function of τ whose period is $T_H = 2\pi/\omega_H$. We transform to cartesian coordinates and expand $\underset{\sim}{G}(\tau)$ as a Fourier series to get

$$G(\tau) = \sum_{k=-\infty}^{\infty} (x_0 G_{xk} + y_0 G_{yk} + z_0 G_{zk})\, e^{ikh_H v_0\tau}, \tag{9}$$

the solution to (6) being also obtained in the form

$$r_v^{(1)}(\tau) = \frac{V_0}{2u_0} \sum_{k=-\infty}^{\infty} (x_0 x_k^{(1)} + y_0 y_k^{(1)} + z_0 z_k^{(1)})\, e^{-ih_k v_0\tau}, \tag{10}$$

in which $h_k = h - kh_H$, $h_H = \omega_H/v_0$, $u_0 = v_0^2/2\gamma_1$ is the voltage, while $x_k^{(1)}$, $y_k^{(1)}$, and $z_k^{(1)}$ are related to G_{xk}, G_{yk}, and G_{zk} by

$$x_k^{(1)} = \frac{G_{xk}}{(h_e - h_k)^2}; \qquad y_k^{(1)} = \frac{(h_e - h_k)G_{yk} + ih_H G_{zk}}{(h_e - h_k)|(h_e - h_k)^2 - h_H^2|}; \tag{11}$$

$$z_k^{(1)} = \frac{(h_e - h_k)\, G_{zk} - ih_H G_{yk}}{(h_e - h_k)|(h_e - h_k)^2 - h_H^2|},$$

in which $h_e = \omega/v_0$.

The general solution to (6) is the sum of the forced motion (10) and the solution $\underline{r}''(\tau)$ to the homogeneous equation corresponding to (6), namely

$$r^{(1)} = |r_V^{(1)}(\tau) + r_{\text{одн}}(\tau)|e^{i\omega t}. \qquad (12)$$

That solution describes the waves in the beam (in the absence of space charge); in the projection on the cartesian system it takes the form

$$\begin{aligned} x_{\text{одн}} &= A_x e^{-ih_e v_0 \tau} + B\tau e^{-ih_e v_0 \tau}; \\ y_{\text{одн}} &= A_y e^{-ih_e v_0 \tau} + Ce^{-i(h_e+h_H) v_0 \tau} + De^{-i(h_e - h_H) v_0 \tau}; \\ z_{\text{одн}} &= A_z e^{-ih_e v_0 \tau} + iCe^{-i(h_e+h_H) v_0 \tau} - iDe^{-i(h_e-h_H) v_0 \tau}, \end{aligned} \qquad (13)$$

in which A, B, C, D are constants of integration.

Now (11) shows that the perturbation produced by the external wave is small, in general: $|r^{(1)}| / \lambda \sim V_0/u_0$. But if

$$h = (h_e + mh_H)(1 + \Delta) \qquad (|\Delta| \ll 1), \qquad (14)$$

we find that the three terms in (10) having k = m, m $\pm$ 1 are much larger than all the others ($x_m^{(1)} \sim \Delta^{-2}$, $y_m^{(1)} \sim z_m^{(1)} \sim y_{m\pm1}^{(1)} \sim z_{m\pm1}^{(1)} \sim \Delta^{-1}$). Those three terms define the resonance interaction with the beam.

3. A thin spiral beam guided by an axial field $\underline{B}_0 = B_0 \underline{z}_0$ gives a solution to (2a) of the form

$$x^{(0)} = x_0 - a\cos(\omega_H \tau); \quad y^{(0)} = y_0 - a\sin(\omega_H \tau); \quad z^{(0)} = v_0 \tau. \qquad (15)$$

The general equation describing the perturbation retains the form of (6), but the field $\underline{E}'$ is taken at the point defined by (15). All formulas (7)-(12) are retained, except that here

$G(\tau) \equiv e'(\tau)$, while the substitutions $x \to z$, $y \to x$, and $z \to y$ are made in (11) and (13).

It is more convenient to use a cylindrical coordinate system for the spiral beam (here $r^{(0)} = a$, $\Theta^{(0)} = \omega_H \tau$, $z^{(0)} = v_0 \tau$). Then (9)-(11) become (with $\underline{r}_0$, $\underline{\Theta}_0$, $\underline{z}_0$ as unit vectors):

$$G(\tau) \equiv e_x(\tau) = \sum_{k=-\infty}^{\infty} (r_0 G_{rk} + \Theta_0 G_{\omega k} + z_0 G_{zk})\, e^{ikh_H v_0 \tau}; \tag{9a}$$

$$r_V^{(1)} = \frac{V_0}{2u_0} \sum_{k=-\infty}^{\infty} \left(r_0 r_k^{(1)} + {}_0 a\, \Theta_k^{(1)} + z_0 z_k^{(1)}\right) e^{-ikh_H v_0 \tau}; \tag{10a}$$

$$z_k^{(1)} = \frac{G_{zk}}{(h_e - h_k)^2}; \qquad r_k^{(1)} = \frac{(h_e - h_k) G_{rk} - i h_H G_{\omega k}}{(h_e - h_k)\left[(h_e - h_k)^2 - h_H^2\right]}; \tag{11a}$$

$$a\Theta_k^{(1)} = \frac{(h_e - h_k)\, G_{\omega k} + i h_H G_{rk}}{(h_e - h_k\left[(h_e - h_k)^2 - h_H^2\right]}.$$

2. Dispersion Equation for a Waveguide Containing a beam Guided by an Axial Magnetic Field or by Crossed Electric and Magnetic Fields

1. The propagation of a wave in a waveguide containing a curvilinear beam is described by a system of equations consisting of equations (13a,b) or (14) of I (for the fields), of equations for the amplitudes of the normal waves (in the form of (12) or (19) of I, and of (2a,b). We shall consider as an example a system falling in the second class.

The equation describing the electromagnetic waves and

the waves in the beam is given by (6) and (14) of I as

$$\frac{d^2\mathbf{r}^{(1)}}{d\tau^2}+2i\omega\frac{d\mathbf{r}^{(1)}}{d\tau}-\omega^2\mathbf{r}^{(1)}+\gamma_1\left[\frac{d\mathbf{r}^{(1)}}{d\tau}+i\omega\mathbf{r}^{(1)},\ \mathbf{B}_0\right]=-\eta\sum_s(V_s^+\mathbf{E}_{s\perp}^+ + V_s^-\mathbf{E}_{s\perp}^-). \tag{16}$$

Here $\mathbf{E}_{s\perp}^{\pm}=(\mathbf{e}^{\pm}+[\mathbf{r}^{(0)}\mathbf{b}^{\pm}])\,e^{\mp ih_0 z^{(0)}(\tau)}$ is the field of the normal wave; it is defined in accordance with (20) of I. The amplitudes $V_s^{\pm}$ may be described by means of the expression

$$V_s^{\pm}=\pm\frac{|I_0|}{N_s}\left\{i\omega\int_{0,\tau_L}^{\tau}\mathbf{r}^{(1)}(\mathbf{E}_{s\perp}^{\pm})^*\,d\tau-\int_{0,\tau_L}^{\tau}\mathbf{E}_s^{\pm}\dot{r}^{(0)}\frac{z^{(1)}}{\dot{z}^{(0)}}+\int_{0,\tau_L}^{\tau}\mathbf{E}_s^{\pm}\mathbf{r}^{(1)}\right\}, \tag{17}$$

in which τ_L is the time of flight (in the zeroth approximation) through the entire interaction space.

Now (16), with (17), may be treated as a homogeneous linear equation in $\mathbf{r}^{(1)}$ in which the quantities are periodic functions of τ having the period $T_H = 2\pi/\omega_H$. We expand the functions as Fourier series and present the solution as

$$\mathbf{r}^{(1)}=\sum_k(\mathbf{x}_0\bar{x}_k^{(1)}+\mathbf{y}_0\bar{y}_k^{(1)}+\mathbf{z}_0\bar{z}_k^{(1)})\,e^{i(\xi+kh_H)v_0\tau},$$

from which it is easy to derive an infinite system of linear homogeneous algebraic equations for $\bar{x}_k^{(1)}$, $\bar{y}_k^{(1)}$, $\bar{z}_k^{(1)}$. . The determinant of this system must be zero if there is to be a solution other than a trivial one, so we get an equation for the characteristic numbers ξ, which define the propagation constants for the normal wave in the waveguide. I shall not give that equation in general form; I take the denominator to be convergent and consider only the case of weak currents, for which we need consider only the interaction with the normal wave to which the beam is synchronized. The limitation is all the more justified by the fact that space charge was neglected even when the initial equations, (16) in particular, were being deduced.

2. The interaction being weak, the propagation constants must be very similar to those for the unperturbed waveguide and beam (h_{0s}, h_e, $h_e \pm h_H$),; then condition (14) can be put in the form

$$(1+\varepsilon)h_0 = (h_e + mh_H) \qquad (|\varepsilon| \ll 1), \tag{18}$$

in which $h_0 = h_{0p}$ is the propagation constant of the p-th normal wave in the unperturbed waveguide. This result can be interpreted as the condition that the phase velocity of the p-th wave must coincide approximately (within an amount ε) with the phase velocity of the spatial harmonic of the beam current*.

Only effects second-order in relation to the small corrections $\delta = (h_i - h_0)/h_0$ and ε are entailed as a result of neglecting terms relating to all waves other than the p-th (synchronous) one during our search for h_i close to $h_{0p} = h_0$. Therefore we neglect all but one wave on the right in (16) in considering a trochoidal beam, when we can represent the solution of that equation in the form (10-(13). Then $\underline{r}_V^{(1)}$, substituted into (17), gives us an approximate dispersion equation (only the synchronous wave is considered). In this connection we need retain only one term $\underline{x}_0(V_0/2U_0)x_m^{(1)}$ in the expression for $\underline{r}(1)$ (provided that $G_{xm} \neq 0$) and we may strike out the last two terms in (17), because those terms are of order $\delta \ll 1$ relative to the first. We integrate with respect to τ and take

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* Provided that the space-charge density is negligible, the three propagation constants for the waves in a beam in a uniform magnetic field differ one from another by h_H, in accordance with (13). The period with which the spatial modulation propagates is $2\pi/h_H$, so the spectrum of the spatial harmonics is threefold degenerate (sixfold, if the space-charge waves are included). Then (19) is complied with simultaneously for the harmonics (m-1,m,m+1) of three (six)waves in the beam; this explains the three resonance terms in (10) - (11) for $\underline{r}_V^{(1)}$.

only the terms of order $1/\delta\ (\delta - \epsilon)^2$ to get the dispersion equation for a waveguide containing a trochoidal beam guided by the crossed fields $E_0 = -y_0 E_0$, $B_0 = B_0 x_0$) in the form

$$\delta(\delta - \epsilon)^3 = -C_m^3 = -\frac{|I_0|}{4U_0}\frac{2h_z}{h_0^3 N}|G_{xm}|^2 = -\frac{|I_0|}{4U_0}\frac{h_z}{h_0}K_m, \quad (19)$$

in which C_m is a gain parameter [7], K_m is the interaction impedance, $\delta h_0 = h - h_0$ is a correction to the propagation constant, G_{xm} is the m - th Fourier coefficient of the periodic function $G_x(\tau) = e'_x e^{iah_c \sin(\omega_H \tau)}$, e'_x is the x component of the amplitude of the effective field of the normal wave (see section 1) and subscript p indicating the type of synchronous wave is omitted.

A spiral beam guided by a field $\underline{B}_0 = \underline{z}_0 B_0$ gives an approximate dispersion equation of the form of (19), but G_{xm} becomes G_{zm}, which is the m-th Fourier coefficient of the function $G_z(\tau) = e'_z(\tau)$ (see section 1). Again, (19) has the form of the dispersion equation for an ordinary traveling-wave tube having a straight velocity-modulated beam. The difference appears only in the numerical values of the G_{xm} (G_{zm} for a spiral beam) or (which is the same) in those of the K_m. But a modulated beam in general provides that (18) is complied with for any phase velocity in the waveguide; in particular, waves can be generated and amplified even if they are of normal speed in a waveguide with smooth walls. The exact nature of the interaction depends on the dispersion shown by the waveguide.

a) Direct dispersion (the phase and group velocities of the electromagnetic wave coincide in direction; $v_p v_g > 0$, i.e. $h_0 N > 0$); in an amplifier we must use a beam whose mean electron speed v_0 coincides in direction with the

phase velocity ($v_o v_p > 0$; $h_e + mh_H \simeq h_o > 0$), whereas in an oscillator (back-wave tube) or regenerative amplifier we must use a beam whose electrons move in the reverse sense ($v_o v_p < 0$; $-h_e + mh_H \simeq h_o > 0$).

b) Reverse dispersion ($v_p v_g < 0$; $h_o N < 0$); the case $v_o v_p > 0$ corresponds to oscillation amplification ($h_e + mh_H \simeq h_o > 0$), whereas $v_o v_p < 0$ ($-h_e + mh_H \simeq h_o > 0$) corresponds to amplification, the amplitude of the traveling wave increasing exponentially along the line*.

The formal similarity of (19) for the interaction (with G_{xm} or $G_{zm} \neq 0$) to the dispersion equation of type O traveling-wave tube enables us to avoid considering separately the properties of oscillators and amplifiers employing trochoidal and spiral beams. In future we shall call that interaction type O, no matter what the type of field used to guide the beam. The formulas for the gain, starting current, etc are derivable from those for the usual type O traveling-wave tube if the gain parameter C is replaced by the parameter C_m defined by (19).

3. In deducing (19) I assumed that the G_{xm} (I consider a trochoidal system only) are different from zero, which implies that only terms of order $1/\delta\,(\delta - \varepsilon)^2$ on the right in (17) need be considered. If $G_{xm} = 0$ we must include terms of order $1/\delta\,(\delta - \varepsilon)$, $\varepsilon/\delta\,(\delta - \varepsilon)^2$, etc; in that case we get an interaction of a rather different type, which we shall call type M. We use (10) and (11), and separate out the resonance terms in $z^{(1)}$ and $y^{(1)}$ (the terms of order $1/\delta\,(\delta - \varepsilon)$); we

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* These features occur also with a straight beam modulated by an electrostatic field periodic in space [4, 5].

substitute for $r^{(1)}$ in (17) and integrate with respect to τ to get

$$\delta(\delta-s)=\frac{|I_0|}{4U_0}\frac{1}{A_0^2N}\frac{h_z}{h_H}\{|G_{ym+1}|^2+|G_{zm+1}|^2-2\mathrm{Im}(G_{zm+1}G^*_{ym+1})+$$
$$+4\mathrm{Im}(G_{zm}G^*_{ym})-|G_{ym-1}|^2-|G_{zm-1}|^2-2\mathrm{Im}(G_{zm-1}G^*_{ym-1})\}, \quad (20)$$

in which Im denotes the imaginary part. The corresponding equation for a spiral beam with $G_{zm} \equiv 0$ is obtained by making the substitutions $G_{yk} \rightarrow G_{\theta k}$, $G_{zk} \rightarrow G_{rk}$ (in which $G_{\theta k}$ and G_{rk} are the Fourier coefficients of the corresponding field components).

Then (20) describes the type M interaction; it is an equation of second order in δ and in form is the same as the dispersion equation of a type M system having a straight unmodulated beam. The sign of the right half of (20) controls the nature of the roots; it contains terms having m-1, m and m + 1 as subscripts, the cause being the threefold degeneracy (as before). This feature makes it impossible to examine the type M interaction without first specifying exactly the structure of the field in the waveguide. However, G_{zk} and G_{yk} decrease as $|k|$ increases, so a fast wave ($h_o \approx h_e + mh_H$, $m < 0$) has an interaction for which the right half of (20) is positive (apart from certain special cases); the corrections δ are real, so there is no amplification*.

4. Interactions types O and M both give dispersion equations ((19) and (20)) containing factors G_m determined by the field $E_z = (e + |r^{(0)}b|)e^{-ihz}$. . At first sight it seems that

* An example of a type M interaction is considered in [8] (a paper at the Third All-Union Conference of MVO on Radio and Electronics, Kiev, 1959).

that there is no need to include the high-frequency magnetic field $\underline{b}$, because $v_p \gg c$ and $|b| \leqslant |e|/c$ in a normal waveguide (c is the velocity of light), so the correction is of order $\underline{r}^{(0)}/c$. But the fact is that the important quantity is not the field $\underline{e} \exp(-ih_0 z)$ itself but rather the variation in that field along the static trajectory of an electron. That variation is of order $\sim ea\omega/c$; , whereas the Lorentz force is $|\dot{r}_0 b| \sim \omega_H a b \sim e a \omega_H / c$. . So, if $\omega \sim \omega_H$ (which is precisely the case of interest), the high-frequency magnetic field in general produces an effect comparable with the effect produced by the electric field**.

3. Type O Interaction in Systems with Spiral and Trochoidal Beams

The dispersion equations for these two types of beam are formally similar, but the mechanisms whereby the electrons interact with the beams are quite different. Some detailed examples form the best means of illustrating the differences.

1. Spiral beam guided by an axial magnetic field

$\underline{B}_0 = \underline{z}_0 B_0$. Here $K_m = 2|G_{zm}|^2/h_0^2 N$, i.e. the impedance is proportional to the square of the m-th Fourier coefficient for the amplitude of the axial component of the field (see section 1):

$$G_{zm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[e_z(\zeta) - a\omega_H b_r(\zeta) \right] e^{-im\zeta} d\zeta, \qquad (21)$$

** A straight beam modulated by an electrostatic field is considered in [4, 5]; in that case the Lorentz force can be neglected on account of the presence of an axial magnetic field, which eliminates transverse motion.

in which $\zeta = \omega_H \tau$, a is the radius of the spiral, and b_r is the radial component of the magnetic induction in a system coaxial with the beam. Figure 2 shows typical examples of such systems.

a) Spiral Beam and Slow TM Wave. Here the effect of the magnetic field (of $a\omega_H b_r$ in (21)) can be neglected. If (18) is complied with, the electrons on every occasion (in the zeroth approximation) enter a stronger field in the same phase. The interaction mechanism can be the same as that in an ordinary type O tube*; the e_z component produces the bunching and draws off the power from the bunches. We see from (19) that the required phase relation applies at all times.

b) Spiral Beam in a TE (or TEM) Wave. Here e_z is absent, so the axial component of the Lorentz force $|\dot{r}^{(0)}b|_z = a\omega_H b_r$ provides the bunching. The power is drawn off as a result of the interaction between the bunches moving in phase along the spiral and the transverse high-frequency field.

Both mechanisms operate together in the more general case in which the TM waves are not slow (Fig. 2c).

2. Trochoidal beam in crossed steady fields $E_0 = -y_0 E_0$, $B_0 = x_0 B_0$. . Here we have $K_m = 2|G_{xm}|/h_0^2 N$, in which

$$G_{xm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e_x(\zeta) + v_0 a h_H \sin\zeta b_z(\zeta) + \qquad (22)$$

$$+ v_0(1 - a h_H \cos\zeta) b_y(\zeta)|\, e^{-i(ah\sin\zeta + m\zeta)}\, d\zeta,$$

* G.I. Rapoport pointed out this possibility to me.

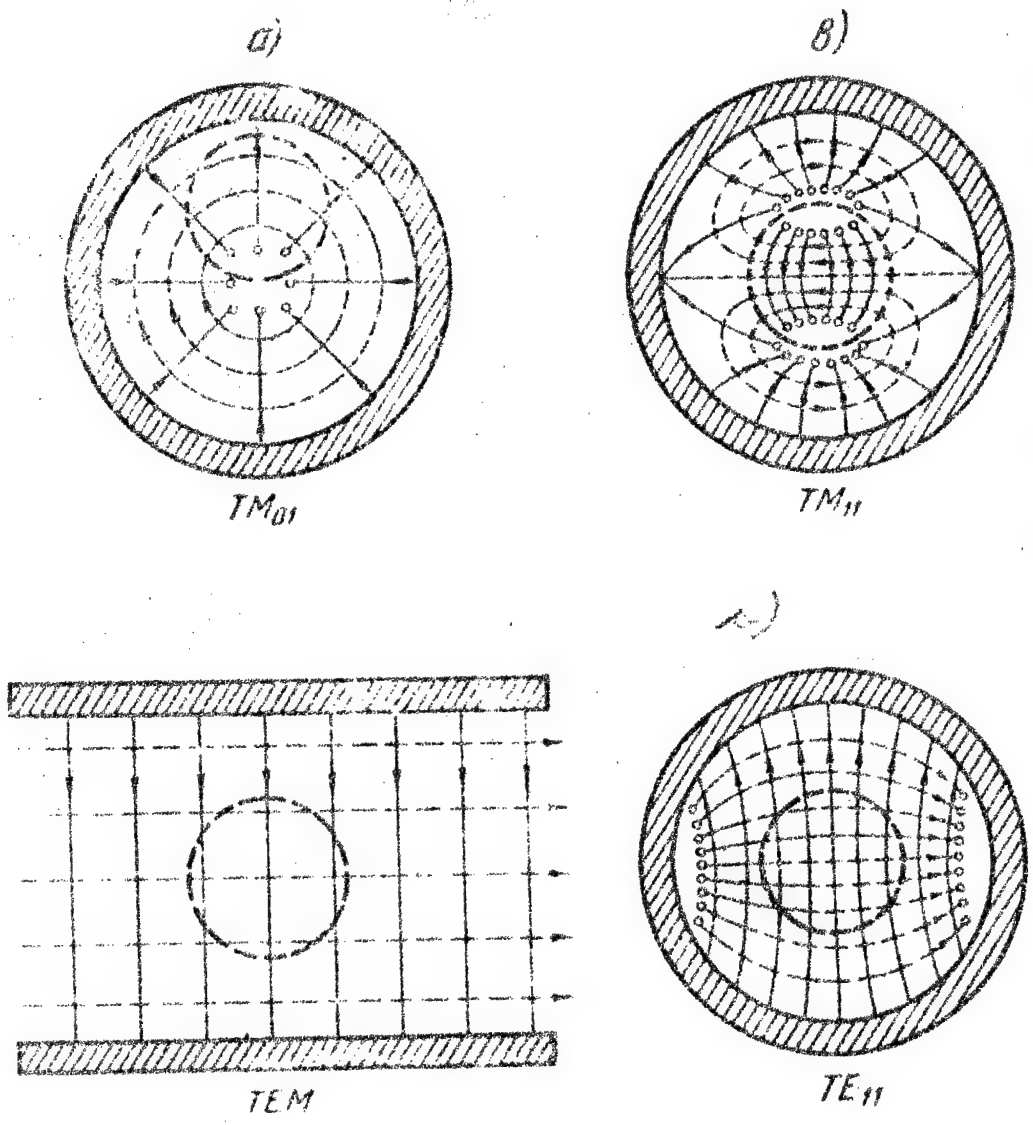

Fig. 2 Systems having spiral electron beams. The broken circles denote the cross-sections of the beams.

and where $\zeta = \omega_H \tau$, a is the amplitude of the oscillations in the trochoid, and $v_o = E_o/B_o$ is the drift speed in the z direction; the interaction impedance is controlled by the transverse component (that parallel to B_o). Figure 3 shows typical examples of systems with trochoidal beams.

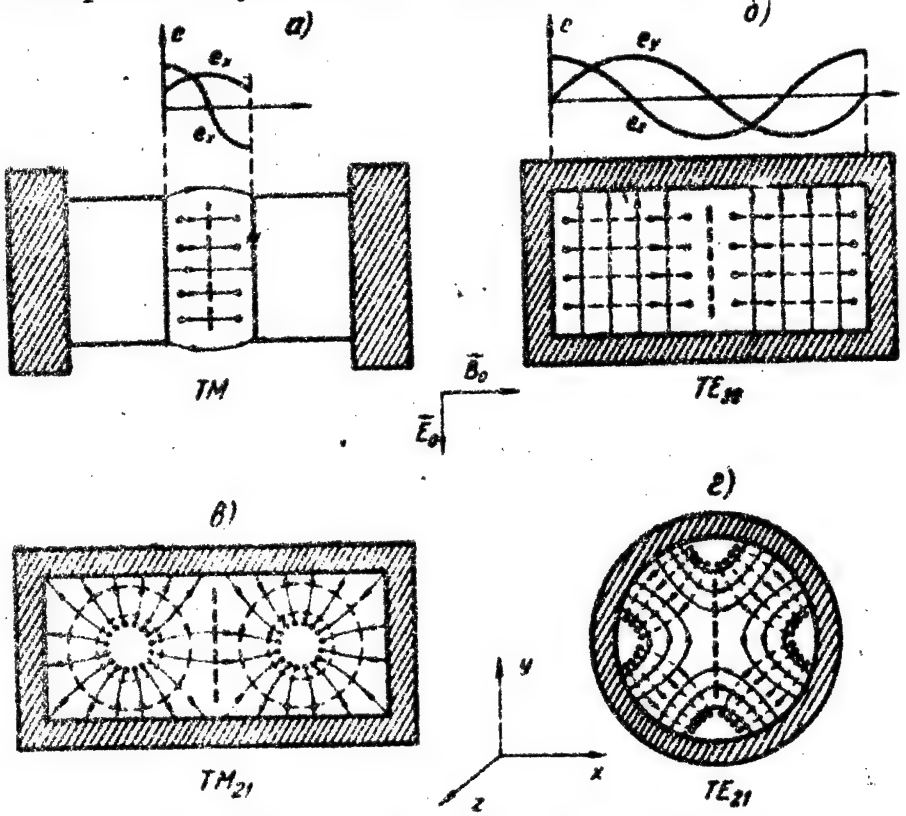

Fig. 3 Systems with trochoidal beams. The broken circles denote the cross-sections of the beams.

a) Trochoidal Beam in a Slow TM Wave. The magnetic field of such a wave is weak, and the Lorentz force appearing in (22) can be neglected. The interaction mechanism is not dependent on axial bunching; it is analogous to that found in a tube having a transverse field [7,9]. The transverse component e_x deflects the beam parallel to $\underline{B}_o$ in a way such that the electrons giving up energy are in a strong part of the axial

field e_z, while those receiving energy are in a weaker part. The unperturbed beam is shown in Fig. 3a in a plane in which $e_z = 0$ (as in a transverse-field tube); this situation is not obligatory.

b) Trochoidal Beam in a TE Wave (Fig. 3b). The Lorentz force deflects the beam in a direction parallel to $\underline{B}_o$; that force derives from the axial high-frequency magnetic field ($e_x \equiv 0$). Most of the electrons oscillating in the yz plane as a result fall into a retarding transverse field e_y, which results in a transfer of energy from the beam to the high-frequency field.

Figure 3 (c and d) shows systems having more complex forms of interaction (combinations of the above simple mechanisms). Figure 3c shows a TM_{21} wave, in which case the beam is deflected by the transverse fields e_x and b_y and the power is drawn off by e_z. Figure 3d shows a TE_{21} mode in circular guide; here e_x produces the deflection in conjunction with b_2 and b_y, while e_y draws off the power.

These examples enable us to evaluate the various beam systems. The main advantage of the spiral and trochoidal beams is that waves of normal speed can be used without resort to periodic structures. This feature simplifies the design of the devices and increases the cross-sectional area of the interaction space to the order of λ^2, which feature enables us to use larger currents and makes it less necessary to focus the beam carefully. The parameter C can be made of the order found for ordinary traveling-wave tubes if the first or second harmonic of the beam current is used, For example, a spiral beam of radius a moving along the axis of a guide whose radius is R and interacting with a T_{11} wave ($h_o = h_e - h_H$ in synchronism) has a gain given by:

$$-\delta^3 = C^3_{TE_{11}} = \frac{|I_0|}{4U_0}\,\frac{\omega\mu}{h_0}\,\frac{h_H^2}{h_e h_0}\,\frac{|\varkappa a J_1'(\varkappa a)|^2}{2\pi\,(\nu_{11}^2-1)\,J_1^2(\nu_{11})},$$

in which $\varkappa = \nu_{11}/R$ is the transverse wave number, ν_{11} being the first root of the derivative of a first-order Bessel function. Taking $\omega_H a \sim 0{,}1c$, $U_0/I_0 \sim 10^4$ ohms, $h_e/h_0 \sim 40$, we have $C_{TE_{11}} \sim 5 \times 10^{-2}$. A spiral beam at the center of a rectangular guide (small side b, wide one d, mode TE_{01}) gives us that

$$-\delta^3 = C^3_{TE_{01}} = \frac{|I_0|}{4U_0}\,\frac{\omega\mu}{h_0}\,\frac{h_H^2}{h_e h_0}\cdot\frac{a^2}{2bd}\,J_0^2\left(\frac{\pi a}{d}\right),$$

which implies that $C_{TE_{01}} \sim 7 \times 10^{-2}$ for $b \sim 2a$ (with the above parameters). It is clear that we get $C \sim 2 \times 10^{-2}$ (which is quite adequate) even if the second harmonic is used ($h_0 = h_e - 2h_H$); in that case J_0^2 in the formula must be replaced by χ^3, in which x_1 is the displacement of the axis of the beam relative to the plane of symmetry.

A trochoidal beam gives a similar result; for example, the system of Fig. 3b has

$$-\delta^3 = C^3_{TE_{02}} = \frac{|I_0|}{4U_0}\,\frac{\omega\mu}{h_0}\,\frac{h_H^2}{h_e h_0}\,\frac{8\pi^2\,|ah_0 J_1(ah_0)|^2}{h_0^4 S_\perp \rho \lambda^2_{кр}},$$

in which $S_\perp$ is the cross-sectional area of the waveguide and λ_c is the cut-off wavelength. Taking $ah_H \sim 1$*, $h_e/h_0 \sim 40$, $U_0/I_0 \sim 10^4$ ohms and $h_0^2 S_\perp \lambda^2_{кр} \sim 50$ we get $C_{TE_{02}} \sim 4 \times 10^{-2}$.

We see that the gain can be quite large for type O

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* The case $ah_H > 1$ (trochoid forming a loop) usually demands a special examination, but the dispersion equation retains the form of (19).

interaction when the beam is modulated by a fixed magnetic field. The linear theory cannot give us the maximum possible output to be expected from such devices, but the mechanisms of interaction with transverse electromagnetic fields are such as to lead us to hope that such devices may have quite high efficiencies.

Conclusion

A study of the interactions between electromagnetic waves in waveguides and spiral or trochoidal electron beams has shown that type O interaction (that corresponding to equation (19)) can be used in microwave amplifiers and oscillators that contain no periodic structures or retarding systems. Type M interaction (that corresponding to equation (20)) in most cases does not produce amplification when one of the fast spatial harmonics of the beam is synchronized with a wave of normal speed. This topic forms the subject of a more detailed paper to appear shortly [8].

However, we should not conclude that high-frequency amplification is impossible simply because the dispersion equation obtained in the linear approximation does not have complex roots. Interactions with weak fields can change the paths followed by the electrons very substantially, especially if the steady and high-frequency fields have large gradients, because it is essential to consider the nonlinear terms in such cases. We have found that a mechanism capable of providing amplification with a wave of normal speed can occur in such systems [10,11], although that result derives from a study made on a simple model by means of a very simple approxiamtion for the nonlinearities (one in which the electrons being accelerated are removed when their energies exceed a certain limit).

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ON THE THEORY OF THE RESONATING LOAD IN VHF GENERATORS

Pages 463-472 //

by V.A. Malyshev

The author examines the operation of a certain class of VHF generator with an additional resonator, coupled with the main resonator via a transmission line, as well as the useful load. The investigation covers the case of symmetrical generation zones and, while neglecting the "line length effect", includes a consideration of the following: the appearance of frequency jumps, the possibility of broadening the band by electronic tuning, the discrimination of electronic tuning and characteristics of the generation zone.

Introduction

Any self-oscillator may be represented by the network shown in Fig. 1a. If $Y = G + jB$ is the admittance of the oscillating system (OS) and $F(u_0, \omega, u)$ is the amplitude of the first harmonic of the current flowing in the circuit of the nonlinear element (NE) of the generator and, in the general case, depending on the parameters of the supply voltage u_0, the frequency ω of the oscillations being generated and the amplitude of the voltage in the oscillating system u, then the action of the generator is defined by the relationships:

$$2P = Gu^2; \quad F(u_0, \omega, u)\cos\delta(u_0, \omega, u) = Gu; \quad B + G\operatorname{tg}\delta(u_0, \omega, u) = 0, \quad (1)$$

where P is the power of the oscillations being generated and

$\delta(u_0, \omega, u)$ is the deviation from π of the phase difference between the current and the voltage in the oscillating system. Usually the

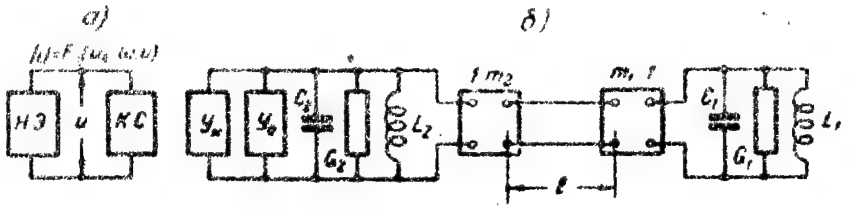

Fig. 1

dependence of $F(u_0, \omega, u)$ on u is called the oscillatory characteristic of the generator and $\delta(u_0, \omega, u)$ the phase of the electronic admittance of the generator. It is possible to show [1] that for a second-order oscillating system the condition of the stability of steady generation has the form

$$\frac{\partial}{\partial u}\left[\frac{F}{u}\cos\delta\right] < 0, \qquad (2)$$

and the self-excitation condition is written:

$$C = \lim_{u \to u_1}\left[\frac{F}{u}\cos\delta\right] \geqslant G, \qquad (3)$$

u_1 being zero for soft excitation and for hard excitation being defined by the value of u for which condition (2) is first violated.

Let us consider a class of VHF generators satisfying the following conditions:

1) The phase of the electronic admittance scarcely changes with variation in the frequency and amplitude of the oscillations being generated (i.e. $\delta(u_0, \omega, u) \simeq \delta(u_0)$). The criterion for the fulfilment of this condition is constancy of the slope of the lines of equal frequency in the load characteristics of the generator.

2) By varying a certain parameter of the supply voltage u_0 in accordance with a certain law it is possible to vary δ in such a way

as to insure symmetry of variation in time of the phase δ with respect to its null value, corresponding to the center of the generation zone. The generator will then run through all points in the generation zone in succession.

3) The generator has a soft excitation regime.

4) The oscillatory characteristic of the generator scarcely changes within the limits of the generation zone, i.e. $F(u_0, \omega, u) \approx F(u)$.

It is possible to show that the conditions stated above are satisfied with good approximation by the reflex klystron, the monotron, certain types of transit klystron generator and a number of other VHF generators.

Here we shall consider the behavior of the class of VHF generator mentioned in the presence of an additional resonator, coupled with the main resonator by means of a lossless transmission line. For this purpose we shall examine the case in which the additional resonator is coupled to the main resonator by means of individual connecting elements and a connecting line (independent connection) and also the case in which the additional resonator is inserted in the high-frequency channel leading to the useful load (intermediate connection). The "line length effect" is not taken into account.

It is known [2] that the equivalent network representation of the resonator of an oscillating system with lumped constants will vary as a function of the choice of the location of the network in the transmission line. For a network in the form of a parallel oscillating circuit included in the transmission line linking the resonators, the equivalent network for independent connection of the additional resonator is illustrated in Fig. 1b. Here the quantities G_1, C_1, L_1 characterize the additional resonance load, G_2, C_2, L_2 the main oscillating system of the generator; m_1 and m_2 are coefficients of admittance transformation from the transmission line to the terminals of the tanks; l is the length of the line between the points at which the equivalent networks of the resonators may be represented in the form of parallel

oscillating circuits (these points are easily obtained from experiments [2,3]). The quantity $Y_H = G_H + jB_H$ characterizes the useful load of the generator; Y_e is the high-frequency electronic admittance, defined by the relationship:

$$Y_e = u^{-1} F(u) e^{-j[\beta(\omega)+\pi]}. \tag{4}$$

We shall introduce the notation:

$$\gamma = Q_{01}/Q_{02} = G_1/Y_0 m_1; \quad Q_{01} = \omega_1 C_1/Y_0 m_1 \tag{5}$$

(Q_{h1} and Q_{01} are the external and self-Qs of the additional resonator; Y_0 is the wave admittance of the transmission line linking the resonators),

$$M = \cos\beta l - \alpha \sin\beta l; \quad N = \gamma \sin\beta l, \tag{6}$$

where

$$\alpha = 2Q_{01}(\omega - \omega_1)/\omega_1; \quad \omega_1 = 1/\sqrt{L_1 C_1};$$
$$\beta = 2\pi/\lambda_g; \quad G_p = (G_2 + G_n)/Y_0 \tag{7}$$

(λ_g is the wavelength in the line).

Then, using transmission line theory, it is possible to write (1) for the case in question in the form

$$2P_n = G_n u^2; \; u/F(u) = \cos\delta/Y_0 |G_p + \gamma m_2 (M^2 + N^2)^{-1}|; \tag{8}$$

$$Y_0 m_2 (M^2 + N^2)^{-1} |M(\alpha\cos\beta l + \sin\beta l) - \gamma N \cos\beta l| + 2C_2 \Delta\omega_2 +$$
$$+ B_n + Y_0 |G_p + \gamma m_2 (M^2 + N^2)^{-1}| \operatorname{tg}\delta = 0, \tag{9}$$

where $\Delta\omega_2 = \omega - \omega_2$; $\omega_2 = 1/\sqrt{L_2 C_2}$; P_n P_H is the power in the useful load.

Expressions (8) and (9) characterize the behavior of the generator in the case in which the form of the function $u^2 = f[u/F(u)]$ is known. For the class of generators in question this function increases monotonically.

1. Operation of generator when βl is close to $n\pi$.

When $\beta l \simeq n\pi$, where n is an integer, the effect of the line

length begins to become apparent, i.e. the line linking the resonators turns into a resonator itself and the generator develops into a triple-tuned system. This case is not investigated below. However, if the case $l \simeq 0$ is considered, assuming that for this case the line is not resonant, then relations (8) and (9) take on the form:

$$u/F(u) = \cos\delta/k;\ 2C_2\left[(1 + Q_{b1}\omega_2/Q_{b2}\omega_1)\omega - (1 + Q_{b1}/Q_{b2})\omega_2\right] + \\ + B_H + k\operatorname{tg}\delta = 0, \tag{10}$$

where

$$k = \omega_2 C_2\left[1/Q_{oH2} + Q_{b1}/Q_{01}Q_{b2}\right].$$

$Q_{b2} = \omega_2 C_2/Y_0 m_2$ is the external Q of the principal resonator and Q_{oH2} is defined by the expression:

$$Q_{oH2} = \omega_2 C_2/(G_H + G_2). \tag{11}$$

Starting condition (3) then has the form:

$$C_0 > k/\cos\delta;\ C_0 = \lim_{u\to 0}\left[F(u)/u\right]. \tag{12}$$

It is possible to show that up to the points at which generation ceases the electronic tuning band is defined by the expression

$$\Delta\omega_p = \frac{\sqrt{C_0^2 - k^2}}{C_2(1 + Q_{b1}\omega_2/Q_{b2}\omega_1)}, \tag{13}$$

and up to points at which the power has diminished n times compared with that in the center of the zone by the expression:

$$\Delta\omega_n = \frac{\sqrt{nF^2(u_m/\sqrt{n}) - F^2(u_m)}}{u_m C_2(1 + Q_{b1}\omega_2/Q_{b2}\omega_1)}. \tag{14}$$

Comparing the expressions obtained with (1) shows that in this case inserting an additional resonator is equivalent to changing the Q and the resonance frequency of the main resonator and proves to have a stabilizing effect on the frequency of the oscillations being generated. The degree of pulling of the frequency by the load $\partial\omega/\partial B_H$ then decreases from insertion of the additional load by a factor S, where the stabilization factor S is given by the expression:

$$S = 1 + Q_{b1}\omega_2/Q_{b2}\omega_1. \tag{15}$$

2. Condition of symmetry of generation zone

Equation (9), defining the frequency of the oscillations being generated, when $E_H = 0$ and with the substitution

$$B = (Q_{01}/Q_{b1})\ (\operatorname{ctg}\beta l - \alpha), \tag{16}$$

from which it follows that

$$M = \gamma B \sin\beta l; \quad N = \gamma \sin\beta l, \tag{17}$$

can be brought to the form:

$$y^3 + 3py + 2q = 0, \tag{18}$$

where

$$y = B - \frac{r\operatorname{tg}\delta - \Omega}{3}; \quad p = \frac{1-z}{3} - \left(\frac{r\operatorname{tg}\delta - \Omega}{3}\right)^2; \tag{19}$$

$$q = \frac{\Omega - r\operatorname{tg}\delta}{3}\left[\left(\frac{r\operatorname{tg}\delta - \Omega}{3}\right)^2 + 1 + \frac{z}{2} + \frac{3z}{2r}\right] - \frac{z\Omega}{2r}, \tag{20}$$

with

$$\Omega = (Q_{01}/\omega_1 Q_{b1} Q_{b2})\ |(Q_{b1}\omega_2 - Q_{b2}\omega_1)\operatorname{ctg}\beta l + 2\ (\omega_2 - \omega_1)\ Q_{b1} Q_{b2}|; \tag{21}$$

$$r = \omega_2 Q_{01}/\omega_1 Q_{02}; \quad z = Q_{01}^2 \omega_2/Q_{b1} Q_{b2} \omega_1 \sin^2\beta l. \tag{22}$$

Within the limits of the generation zone the generator will operate without frequency and power jumps, if the self-excitation condition

$$C_0 \cos\delta > Y_0\ |G_p + \gamma m_2\ (M^2 + N^2)^{-1}| \tag{23}$$

and the stability condition are fulfilled for only one of the roots of equation (18). Therefore, a sufficient condition of generation without jumps will be the condition of the existence of a single real root of equation (18):

$$q^2 + p^3 > 0. \tag{24}$$

It follows from (8) and (17) that the generation zone will be symmetrical with respect to the center, if the function $B = f(\delta)$, defined by eq. (18), is symmetrical or antisymmetrical with respect to the

point $\delta = 0$. It follows from (18)-(21) that the latter condition will be satisfied when $\Omega = 0$. Therefore Ω may be called the parameter of asymmetry of the zone. The condition $\Omega = 0$ may be rewritten in the form:

$$\lambda_1 = \lambda_2 \frac{Q_{b2}(\operatorname{ctg}\beta l + 2Q_{b1})}{Q_{b1}(\operatorname{ctg}\beta l + 2Q_{b2})} \tag{25}$$

(λ_1 and λ_2 are the resonance wavelengths of the additional and main resonators) and constitutes the condition for getting symmetrical generation zones, which are of some importance in practice.

Henceforth condition (25) will always be assumed to be satisfied.

3. Operation of generator for symmetrical zones

Let us assume that $\beta l \neq n\pi$ and $\Omega = 0$. Then condition (24) assumes the form:

$$t(r\operatorname{tg}\delta)^4 + l(3r\operatorname{tg}\delta)^2 + 3(1-z)^2 \geqslant 0, \tag{26}$$

where

$$t = 3(1 + z/r);\ 4l = [t^2 + 2t(z-1) - (1-z)^2/3].$$

In the case of strong coupling ($z > 1$) (26) is not fulfilled in the center of the zone, where

$$\operatorname{tg}^2\delta \leqslant \frac{9}{2r^2t}\left(\sqrt{t^2 - \frac{4t(1-z)^2}{27}} - l\right). \tag{27}$$

In this region (18) has three roots

$$y_1 = 2\sqrt{b}\,\frac{\operatorname{tg}\delta}{|\operatorname{tg}\delta|}\cos\frac{\theta}{3}; \tag{28a}$$

$$y_2 = -2\sqrt{b}\,\frac{\operatorname{tg}\delta}{|\operatorname{tg}\delta|}\cos\frac{\pi-\theta}{3}; \tag{28b}$$

$$y_3 = -2\sqrt{b}\,\frac{\operatorname{tg}\delta}{|\operatorname{tg}\delta|}\cos\frac{\pi+\theta}{3}, \tag{28c}$$

where

$$b = \frac{z-1}{3} + \frac{r^2}{9}\operatorname{tg}^2\delta;\ \cos\theta = \frac{r|\operatorname{tg}\delta|}{3b\sqrt{b}}\left(\frac{r^2\operatorname{tg}^2\delta}{9} + 1 + \frac{z}{2} + \frac{3z}{2r}\right). \tag{29}$$

It is possible to show [4] that the root y_3 always corresponds to an unstable state. For the region in which (27) is not satisfied, the frequency of the oscillations is characterized by the root

$$y = 2\sqrt{b}\,\frac{\operatorname{tg}\delta}{|\operatorname{tg}\delta|}\operatorname{ch}\frac{\Theta_0}{3}, \tag{30}$$

where $\operatorname{ch}\Theta_0$ is defined by the right-hand member of the second expression in (29).

The general trend of the function $y = f(\tan\delta)$ is shown in Fig. 2a. At the edges of the region of jumps

$$y_0 = 2\sqrt{b}, \tag{31}$$

where $\tan\delta$ in expression (29) for b is defined by the equality (27). If the conditions of stability of generation at the frequency at which generation takes place are violated at the edge of the region of jumps, the generator experiences a wave jump which may be determined from the expression, obtained on the basis of (31)

$$\Delta\lambda = 3\lambda_1\sqrt{b}/2Q_{01}. \tag{32a}$$

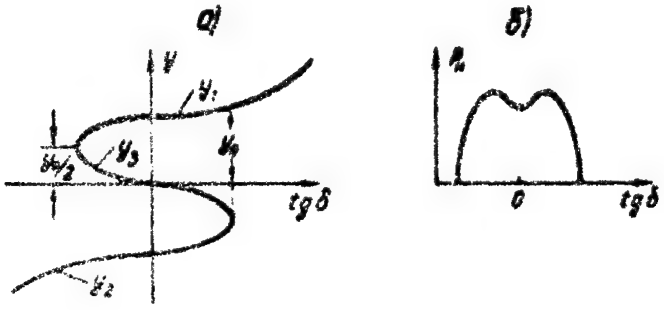

Fig. 2

However, if the stability conditions are violated in the center of the zone, then the jump in wavelength is determined by

$$\Delta\lambda = \lambda_1\sqrt{z-1}/Q_{01}. \tag{32b}$$

Thus, it is possible to have jumps in wavelength lying between the values determined by expressions (32a) and (32b).

In the case of weak coupling ($z < 1$) (26) may not be satisfied only for $l < 0$. However, since $z < 1$, it is easy to show that everywhere $l > 0$ (i.e., when $z < 1$, there are no frequency jumps). The parameter z characterizes the coupling between the resonators. When $z < 1$, for $\tan^2 \delta \leqslant 3(1 - z)\, r^{-2}$ the frequency of the oscillations being generated is determined from the root

$$y = 2\sqrt{-b}\,\frac{\operatorname{tg}\delta}{|\operatorname{tg}\delta|}\operatorname{sh}\frac{\Theta_1}{3};\quad \operatorname{sh}\Theta_1 = -\frac{r\operatorname{tg}\delta}{3b\sqrt{-b}}\left(\frac{r^2\operatorname{tg}^2\delta}{9} + 1 + \frac{z}{2} + \frac{3z}{2r}\right), \tag{33}$$

and for larger values of $\tan^2 \delta$ the frequency is determined from the root (30).

Since in the center of the zone the condition B = 0 must be satisfied, the wavelength λ_0 of the oscillations generated in the center of the zone is determined by

$$\lambda_1 - \lambda_0 = (\lambda_0/2Q_{01})\operatorname{ctg}\beta l. \tag{34}$$

When $\Omega = 0$, expression (8) takes the form:

$$D = \frac{u}{F(u)}(G_n + G_z) = \frac{(1 + B^2)\cos\delta}{(1 + B^2) + z/r}. \tag{35}$$

Since B depends on δ, with B = 0 when $\delta = 0$, the function u/F(u) = $= f(\delta)$ may decrease as the center of the zone is approached. This leads to a depression at the center of the zone (Fig. 2b); the size of this depression will increase with increase in z/r. For weak coupling ($z < 1$) the condition of generation in the center of the zone has the form

$$C_0 \geqslant Y_0\, G_p\,(1 + z/r), \tag{36}$$

and for strong coupling ($z > 1$)

$$C_0 \geqslant Y_0 G_p\,(1 + 1/r). \tag{37}$$

Solving equations (18) and (35) simultaneously, it is possible to get a quadratic equation defining the connection between D and B :

$$A^2 + [r^2(1 - D^{-2}) - (2z + 1)]A + z(2r + z + 2) = 0, \qquad (38)$$

where $A = 1 + B^2$. Since at the points of maximum power the relationship $\partial D/\partial A = 0$ is satisfied, the equality

$$A^2 = (1 + B^2)^2 = z(2r + z + 2), \qquad (39)$$

holds true for these points, the power being determined by

$$P_n = \frac{G_n u^2}{2}; \quad \frac{u}{F(u)} = \frac{r}{Y_0 G_p \sqrt{r^2 - (2z + 1) + 2\sqrt{z(2r + z + 2)}}}. \qquad (40)$$

If the equality (39) is observed for $B = 0$, then the generation zone has a maximally flat top. Therefore the condition for a flat top in the generation zone has the form:

$$r = \frac{1}{2z} - \frac{z}{2} - 1 \qquad (41)$$

and can be realized when $z < \sqrt{2} - 1$, the power generated at the center of the zone then being determined by the condition

$$\frac{u}{F(u)} = \frac{1}{Y_0 G_p} \frac{1 - 2z - z^2}{1 - 2z + z^2}. \qquad (42)$$

It is possible to show that up to points in the zone defined by the value of D the electronic tuning band is found from the expression

$$\Delta\omega = (\omega_1/Q_{01})\sqrt{A - 1} =$$

$$= (\omega_1/Q_{01})\sqrt{z - 1 + 2^{-1}\left[a + \sqrt{a^2 + 4z(a - 2r - 2)}\right]}; \qquad (43)$$

$$a = 1 + r^2(D^{-2} - 1).$$

If we consider the electronic tuning band up to the points at which generation dies away, then in (43) we must put

$$D = G_p Y_0 / C_0, \qquad (44)$$

and for points of the zone, at which the useful power is n times less than at the maximum of the zone (where $u = u_m$), we must substitute in (43)

$$D = Y_0 G_p u_m / \sqrt{n}\, F(u_m/\sqrt{n}), \qquad (45)$$

the quantity u_m being found from the second expression in (40).

4. Electronic frequency tuning

Investigation of the expression under the root sign in formula (43) shows that the function A(z) has a maximum in the point

$$z = \frac{r(r+1)\,|D^2(1+r) - r|}{r^2 - D^2(r+1)^2}. \qquad (46)$$

Since $z > 0$, condition (46) is fulfilled when

$$\frac{r}{1+r} > D^2 > \frac{r^2}{(1+r)^2}. \qquad (47)$$

If we consider the region $0 < z < 1$, then the lower limit of variation of D^2 in expression (47) should become greater so that the condition for a maximum of the function A(z) in the region $0 < z < 1$ will have the form:

$$r(1+r)^{-1} > D^2 > r^2(r+2)(r+1)^{-2}. \qquad (48)$$

Thus, the function A(z) will increase with increase in z in the region $0 < z < 1$, if

$$D^2 < r^2(r+2)(r+1)^{-2}, \qquad (49)$$

and will decrease when

$$D^2 > r/(1+r). \qquad (50)$$

The behavior of the function A(z) characterizes the change in the quantity $\Delta\omega$ in (43) for change in z. Where condition (49) is satisfied, the maximum band will be observed for the critical coupling $(z = 1)$* and in the case of condition (50) for $z = 0$.

From (1) and the network shown in Fig. 1b it is possible to find that in the absence of an additional resonator the electronic tuning band is defined up to points determined by the quantity D by the

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* The condition $z = 1$, necessary for maximum broadening of the band, was first obtained by another process of reasoning, set forth by the author in a paper, presented at the II All-Union Conference of the MVO on Radio Electronics.

relationship:

$$\Delta \omega_0 = \omega_1 Q_{01}^{-1} r \sqrt{D^{-2} - 1}. \qquad (51)$$

Let us consider the relative variation in the electronic tuning $\Delta\omega/\Delta\omega_0$ up to points in the zone, determined by the quantity D, with the additional resonator inserted. Investigation shows that when (48) is satisfied the points (46) correspond to the maxima of the function $\Delta\omega/\Delta\omega_0 = f(z)$; these maxima lie on the curve

$$\frac{\Delta\omega}{\Delta\omega_0} = \sqrt{\frac{(z+r)(z+r+1)}{r^2+2rz+r+z}}. \qquad (52)$$

The greatest value of the right-hand side of this expression is $\sqrt{2}$ and is realized for z = 1, r = 0. When (49) is observed, the greatest value of $\Delta\omega/\Delta\omega_0$ occurs when z = 1. Investigation of the case z = 1 shows that the function $\Delta\omega/\Delta\omega_0 = f(r)$ increases from the values determined by the curve*

$$\Delta\omega/\Delta\omega_0 = \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{2\sqrt{2r+3}-3}}, \qquad (53)$$

to maxima, lying on the line

$$\frac{\Delta\omega}{\Delta\omega_0} = \left[\frac{4(3r+2)}{9r+6-\sqrt{9r^2+20r+12}}-1\right]^{1/2}, \qquad (54)$$

while on this line the equality

$$D = 2r\left(9r+4r^2+6-\sqrt{9r^2+20r+12}\right)^{-1/2}. \qquad (54a)$$

is observed.

For further increase in r the function $\Delta\omega/\Delta\omega_0 = f(r)$ decreases tending to unity. Thus, the greatest relative broadening of the electronic tuning band up to points of the zone with invariable D (for independent connection of the additional resonator these are points with an invariable level of useful power), determined by (54) when (49)

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* On this curve there is satisfied the condition

$$D = r\left(2\sqrt{2r+3}-3+r^2\right)^{1/2}$$

which corresponds, as it is easy to see from (40), to points of maximum power.

is satisfied, occurs when

$$\left(\frac{\Delta\omega}{\Delta\omega_0}\right)_{\max} = \sqrt{\frac{8}{6-\sqrt{12}} - 1} = 1{,}465. \qquad (55)$$

When (50) is observed, the greatest value of the ratio $\Delta\omega/\Delta\omega_0$ equal to unity, will be for z = 0; consequently, in this case the additional resonator decreases the electronic tuning band. This decrease is greatest for z = 1 and is determined by expression (53). (It is possible to show that on curve (53) condition (50) is satisfied and that on curve (34) condition (49) is satisfied.)

The derivative of the electronic tuning in the center of the zone is an important parameter. From (18) it is possible to find that for $z < 1$ this quantity is determined by the expression

$$d\omega/d(\operatorname{tg}\delta)_0 = -\omega_1(z+r)/2Q_{01}(1-z); \qquad (56)$$

its maximum constancy in the neighborhood of the center of the zone being determined from the condition $d^3\omega/d(\tan\delta)_0^3 = 0$ and being realized for z = 0, when

$$d\omega/d(\operatorname{tg}\delta)_0 = -\omega_1 r/2Q_{01}. \qquad (57)$$

For independent connection of the additional resonator it is convenient to introduce, in connection with the two-humped form of the zone (Fig. 2b), the idea of a band of electronic frequency tuning up to points of the zone, having a useful power identical with the center of the zone ($\Delta\omega_c$), and of a band of electronic frequency tuning up to points of maximum power ($\Delta\omega_m$). It follows from expressions (35), (40) and (43) that these quantities are defined by the relationships:

$$\Delta\omega_c = \frac{\omega_1}{Q_{01}}\left|z(2r+z+2)-1\right|^{1/2}, \qquad (58)$$

$$\Delta\omega_m = \frac{\omega_1}{Q_{01}}\left|\sqrt{z(2r+z+2)}-1\right|^{1/2}. \qquad (59)$$

5. Intermediate connection of additional resonator

The network of Fig. 1b and the above reasoning apply to the case of independent connection of the additional resonator. For intermediate connection it is necessary that the quantity Y_H in the network of Fig. 1b be transferred into the circuit of the additional resonator. This changes the significance of the quantities G_p, Q_{OH2} and Q_{O1}, which entered into all the previous calculations, and they now get the values:

$$G_p = G_2/Y_0;\ Q_{01} = \omega_1 C_1/(G_1 + G_n);\ Q_{0\omega 2} = \omega_2 C_2/G_2. \qquad (60)$$

If we now take into account that the useful power, previously determined from (8) and (30), will now be given by

$$P_n = \frac{Q_{01}^2 m_2}{2Q_{01}^2 m_1 \sin^2 \beta l} \frac{G_n u^2}{(1+B^2)}, \qquad (61)$$

then all the expressions obtained above remain valid. The exceptions are formulas (39)-(42), (45), (58) and (59), which account for specific features of the generation zone. It follows from (61) that for intermediate connection the two-humped form of the generation zone may turn into a three-humped form for certain aspects of the oscillatory characteristic (P_H is non-uniquely determined by the quantity u and might increase with approach to the center of the generation zone, when $B \to 0$, although the value of u may then be constant or even decrease slightly).

The conclusions of section 4 all remain in force for the case of intermediate connection. However, we should bear in mind that the quantity D, which for independent connection uniquely determined the useful power, for intermediate connection determines only the magnitude of the high-frequency voltage u in the main resonator.

This investigation makes it possible to estimate characteristics of the operation of generators in the class in question in the presence of a resonating load and to utilize these characteristics in practice. Thus, if it is necessary to achieve maximum broadening of the band of

electronic frequency tuning up to points in the zone, determined by the quantity D (for a known function F(u)), then in cold measurements we must choose values of the quantities Q_{o1}, Q_{b1}, Q_{b2}, λ_1, λ_2 and βl such that condition (54a) is satisfied and equality (25) is fulfilled for z = 1.

For a reflex klystron the oscillatory characteristic has the form:

$$F(u) = 2M I_0 J_1(x), \tag{62}$$

where M is the coefficient of electronic interaction, I_o is the current feeding the resonator gap, x is the bunching parameter and $J_1(x)$ is a Bessel function of the first order. This case has been investigated qualitatively in [5] by employing an equivalent network in the form of two coupled inductive circuits. Such a network is obtained as a particular case of the network if Fig. 1b for $\beta l = (2n + 1)\pi/2$ (n = 0,1,2,3,....).

The relationships obtained in sections 1-5 show that the operation of a generator is determined by four parameters: the coupling parameter z, the zonal asymmetry parameter Ω, the load parameter r and the parameter D which depends on the electronics of the generator. The first three parameters may easily be gotten from cold measurements, which is a considerable convenience in practice.

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THE EFFECTS OF A LARGE-AMPLITUDE RADIOFREQUENCY PULSE ON A SELF-EXCITED OSCILLATOR

Pages 473-482

By M.K. Chirkov & P.N. Zanadvorov

A discussion is presented on the transients arising when a large-amplitude radiofrequency pulse acts on a self-excited oscillator. Use is made of a method derived from the theory of periodic phase locking* and of the properties of finite-difference equations. The resulting analytic expressions agree well with the results from a numerical integration of the differential equations for the problem.

Much is known about the transients in a self-excited oscillator subject to small external perturbations, but less is known about the equally interesting case in which the perturbations are large. The cause is that the method usually employed (asymptotic expansion in terms of powers of a small parameter) here gives rise to a set of equations not soluble in quadratures. It is less convenient to solve that set of equations numerically for each particular case [1] than to consider an analytic solution, because only the latter can give us expressions for the transients in general form.

* By periodic phase locking is meant the forced specification of the initial phase of the oscillator as a result of a periodically applied pulse. Synchronization is a different concept, because it does not imply inherent stability and because it includes within itself the transients produced. The mathematical formulation of the periodic phase locking process is the same as the point-transform method.

The oscillator to be considered here is a tube having a resonant circuit in the plate circuit; the bias (which can be of various types) is greater than the cut-off value. A piecewise-linear curve (open polygon) is used to approximate the tube characteristic.

A method derived from the theory of automatic control [3] has been used to solve the steady-state problem for an oscillator acted on by a sine-wave signal. That method enables us to derive resonance curves and to delineate the stability regions. The results for the steady state are not presented, because they agree exactly with those given elsewhere [1].

The transients produced by a large-amplitude pulse are examined by means of the phasing-function method [2].

1. Formulation and Deduction of the Basic Equations

The instantaneous amplitude and phase of a self-excited oscillator are represented by the equations

$$a(t) = a(t, \varphi_0, a_0, \xi_i);$$

$$\varphi(t) = \varphi(t, \varphi_0, a_0, \xi_i),$$

in which the ξ_i are various parameters (of the perturbation, of the circuit, and of the feedback loop).

As those instantaneous quantities we take the polar coordinates of the image point on the phase locus, the axes of the phase plane being taken as x and ωx for that purpose.

The first cycle of the pulse produces a transient that alters the initial phase and instantaneous amplitude, so the next cycle finds the oscillator in a new state with new values of those parameters, which it alters again, and so on. Then n cycles produce a sequence of n instantaneous phases and amplitudes (each one in the sequence corresponds to the start of its appropriate cycle), with the result that a stationary phase is forced upon the oscillator.

The instantaneous phase and amplitude at the start of

each fresh cycle are determined solely by their values at the start of the previous cycle if the perturbation has a fixed frequency and if the ξ_i are constant; so $a_n = F(\varphi_{n-1}, a_{n-1})$; $\varphi_n = \Phi(\varphi_{n-1}, a_{n-1})$, in which a_n and φ_n are those instantaneous quantities at the end of the n-th cycle, and a_{n-1} and φ_{n-1} are the same at the end of the (n-1)th cycle. Here n is a discrete variable, so we have a set of finite-difference equations

$$\begin{aligned} a(n) &= F[\varphi(n-1),\ a(n-1)]; \\ \varphi(n) &= \Phi[\varphi(n-1),\ a(n-1)]. \end{aligned} \qquad (1)$$

We shall not consider the definitions of stability and state of equilibrium, because those definitions are obtainable from a known method [2].

System (1) must be formulated and solved if the transients are to be found; a method used to find the transients in automatic control systems [3] may be used to formulate system (1).

Let the oscillation at time t = 0 take the form

$$U_n(t) = a_0 A_0 e^{-\delta t} \sin(\omega_0 t + \varphi_0),$$

in which a_0 is the ratio of the initial amplitude to A_0, the amplitude of the free oscillations; then that method gives (if we omit some simple steps) that

$$a_1 = a_0 e^{-\vartheta} \sqrt{1 + 2\frac{\sin\theta_1}{a_0 \sin\theta_0}\sin(\omega_0 t_1 + \varphi_0) + \vartheta^2 \frac{\sin^2\theta_1}{a_0^2 \sin^2\theta_0}}; \qquad (2)$$

$$\varphi_1 = \operatorname{arc\,tg}\frac{\sin\varphi_0 + (\vartheta \sin\theta_1 / a_0 \sin\theta_0)\cos\omega_0 t_1}{\cos\varphi_0 + (\vartheta \sin\theta_1 / a_0 \sin\theta_0)\sin\omega_0 t_1} + b\vartheta,$$

in which ϑ is the logarithmic decrement of the resonant circuit, $b = (\omega_0 - \omega)/\vartheta$ is the generalized mistuning, θ_0 is the angle at which the plate current is cut off during free oscillations ($\sin\theta_0$ is replaced by 1 if $\theta_0 > 90^\circ$), θ_1 is that angle when the first cycle is operating ($\sin\theta_1$ is replaced by 1 if

$\theta_1 > 90^\circ$), and t_1 is the time at which the highest voltage appears on the grid during the first cycle.

Let the external signal at the grid take the form

$$E(t) = \beta A_0 \sin \omega t,$$

in which β is the ratio of the amplitude of that signal to A_0. Then t_1 is given by

$$a_0 \sin(\omega_0 t + \varphi_0) + \beta \sin \omega t = \text{макс.} \qquad (3)$$

A self-excited oscillator with a fixed bias*

$$e_c = -e_0 - e_з,$$

(in which e' is the ratio of the cut-off bias to A_0) gives us for sin θ_1 that

$$\begin{aligned} &\sin\theta_1 = \sqrt{(b_1^2 - e_0^2)/b_1^2} && (e_0 \geqslant 0); \\ &\sin\theta_1 \to 1 && (e_0 < 0). \end{aligned} \qquad (4)$$

Here b_1 is the ratio of the maximum voltage to A_0, the arrow denoting that sin θ_1 must be replaced in (2) by 1.

Self-bias, appearing without a lag, gives us that

$$\begin{aligned} &\sin\theta_1 = \sqrt{1 - \left[\cos\theta_c + \frac{1}{b_1}(\cos\theta_0 - \cos\theta_c)\right]^2} && \left(b_1 \geqslant 1 - \frac{\cos\theta_0}{\cos\theta_c}\right); \\ &\sin\theta_1 \to 1 && \left(b_1 < 1 - \frac{\cos\theta_0}{\cos\theta_c}\right), \end{aligned} \qquad (5)$$

in which θ_g is the angle at which the grid current is cut off.

It is easy to transform (2) to the form

$$\begin{aligned} a_1 e^{\delta} \sin(\varphi_1 - b\vartheta) &= a_0 \sin\varphi_0 + \frac{\vartheta \sin\theta_1}{\sin\theta_0} \cos\omega_0 t_1; \\ a_1 e^{\delta} \cos(\varphi_1 - b\vartheta) &= a_0 \cos\varphi_0 + \frac{\vartheta \sin\theta_1}{\sin\theta_0} \sin\omega_0 t_1. \end{aligned} \qquad (6)$$

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* The bias is given in relative units, the unit being A_0.

We shall use (6), in conjunction with (3)-(5), as the basis of our future discussion.

2. External Signal Equal in Frequency to the Natural Frequency of the Oscillator

The main point of interest here is the way in which φ_n is reached. We introduce new variables into (6), namely

$$\zeta_n = a_n \sin(\varphi_n - b\vartheta); \quad \eta_n = a_n \cos(\varphi_n - b\vartheta); \tag{7}$$

the old variables being related to the new by

$$\operatorname{tg}(\varphi_n - b\vartheta) = \zeta_n/\eta_n; \quad a_n^2 = \zeta_n^2 + \eta_n^2.$$

Now, if b = 0, (6) taken with (3)-(5), which give us the variables after the n-th cycle in terms of those after the (n-1)th cycle, takes the form

$$\begin{aligned} \zeta_n &= B(\zeta_{n-1}, \eta_{n-1})\zeta_{n-1}; \\ \eta_n &= B(\zeta_{n-1}, \eta_{n-1})\eta_{n-1} + C(\zeta_{n-1}, \eta_{n-1}) \end{aligned} \tag{8},$$

the initial conditions being $\zeta_0 = \sin\varphi_0$ и $\eta_0 = \cos\varphi_0$, т. е. $a_n = 1$.

It is readily demonstrated that B and C are functions that vary very little with their arguments ζ_{n-1} and η_{n-1}, if $\beta Q \geqslant 20$ (Q being the Q of the resonant circuit), we can replace them by constants. The approximation is the better the greater the external signal, but the restriction $\beta Q \geqslant 20$ is very weak.

Then (8) becomes

$$\begin{aligned} \zeta_n &= B\zeta_{n-1}; \\ \eta_n &= B\eta_{n-1} + C, \end{aligned} \tag{9}$$

in which $B = \exp(-\vartheta) + C/\beta$, in which C is given for a fixed bias $e_c = -e_0 - e'$ $(e_0 \gg 0)$ by

$$C = 8\beta e^{-\theta}\sqrt{1+\beta^2-e_0^2}/(1+\beta^2)\sin\theta_0 ;$$

or (if $e_0 < 0$, and for self-bias) by

when $\theta_0 > 90^\circ$ and $\sqrt{1+\beta^2} < 1-\cos\theta_0/\cos\theta_c$

$$C = 8\beta e^{-\theta}/\sqrt{1+\beta^2} ;$$

for self-bias with $\theta_0 > 90^\circ$ and $\theta_0 > 90^\circ$ и $\sqrt{1+\beta^2} \geqslant 1-\cos\theta_0/\cos\theta_c$ by $C = 8\beta e^{-\theta}\sqrt{1-\left[\cos\theta_c+(\cos\theta_0-\cos\theta_c)/\sqrt{1+\beta^2}\right]^2}/\sqrt{1+\beta^2}$;

and for self-bias with $\theta_0 \leqslant 90^\circ$ by

$$C = 8\beta e^{-\theta}\sqrt{1-\left[\cos\theta_c+(\cos\theta_0-\cos\theta_c)/\sqrt{1+\beta^2}\right]^2}/\sqrt{1+\beta^2}\sin\theta_0 .$$

System (9), solved in accordance with the initial conditions, gives us that

$$\zeta_n = B^n \sin\varphi_0 ;$$

$$\eta_n = B^n\cos\varphi_0 + C\sum_{k=0}^{n-1} B^k ,$$

which transforms to

$$\operatorname{tg}\varphi_n = \frac{B^n \sin\varphi_0}{B^n\cos\varphi_0 + D(1-B^n)}, \qquad (10)$$

in which D = C/(1-B).

Now (10) is the phasing function for a large-emplitude radiofrequency pulse containing n cycles. Figure 1a shows that the main feature of that function is that the point of equilibrium, which corresponds to $\pm\pi$, vanishes when

$$n > \ln[D/(1+D)]/\ln B .$$

Solving (10) for n, we have

$$n = \frac{1}{\ln B}\ln\frac{D\operatorname{tg}\varphi_n}{\sin\varphi_0+\operatorname{tg}\varphi_n(D-\cos\varphi_0)} . \qquad (11)$$

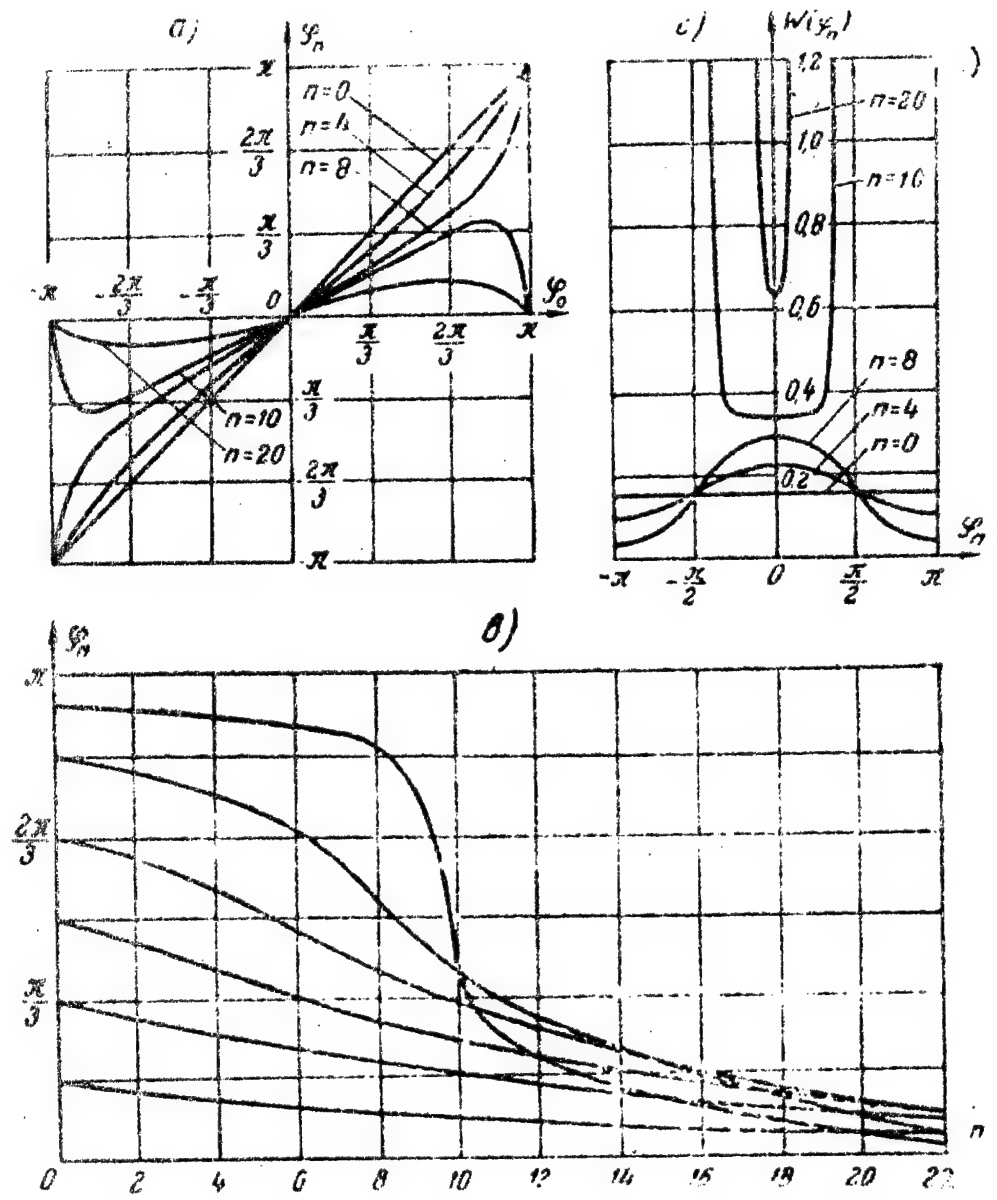

Fig. 1. Mistuning absent ($b = 0$; $\beta = 2.2$; $\delta = 0.1$; $\sin\theta_0 = 1$; $\varepsilon_0 = 0$): a) Phasing function b) statistical distribution of phase, c) phase after pulse has ended

Figure 1c shows the phase ϕ_n (the phase when the pulse has ended) as a function of pulse length for various values of the initial phase. The curves come close together or even intersect as n increases, i.e. from a certain n onwards a large initial phase results in a faster approach to the steady state.

The statistical-distribution function for the phase (if the phase is a random quantity having all its values equally probable) is

$$W(\varphi_n) = (1/2\pi)\,(d\varphi_0/d\varphi_n).$$

Then (10) gives us that

$$W(\varphi_n) = \frac{\sqrt{B^{2n} - D^2(1-B^n)^2\sin^2\varphi_n} + D(1-B^n)\cos\varphi_n}{2\pi\sqrt{B^{2n} - D^2(1-B^n)^2\sin^2\varphi_n}}. \qquad (12)$$

Figure 1b shows the statistical-distribution functions corresponding to periods after pulses of various lengths.

Let us find the pulse length needed to obtain a specified approach to the final phase from a random phase. That approach is specified as the mean deviation $\delta_m\phi$, which is defined (as usual) from the condition that the probability of finding the phase in a range $2\,\delta_m\phi$ is ½. This gives us that

$$\int_{-\delta_{cp}\varphi}^{\delta_{cp}\varphi} W(\varphi_n)\,d\varphi = \frac{1}{2},$$

whence (12) gives us that

$$n = \frac{1}{\ln B}\ln\frac{D\,\mathrm{tg}\,\delta_{cp}\varphi}{1 + D\,\mathrm{tg}\,\delta_{cp}\varphi} \qquad (13)$$

or, if we take $\delta_m\phi$ to be small, that

$$n = \frac{1}{\ln B} \ln \frac{D\delta_{cp\varphi}}{1 + D\delta_{cp\varphi}} . \qquad (14)$$

If we measure the pulse length as $n\vartheta$, then the $n\vartheta$ needed to give the specified $\delta_m\phi$ is almost independent of ϑ.

Figure 2 shows the pulse length needed to give the specified $\delta_m\phi$ as a function of pulse amplitude a) for fixed bias and b) for self-bias. The function clearly takes different forms. Fixed bias implies a pulse length that decreases steadily as β increases and that tends to a limit

$$\vartheta n_{min} = -\ln \frac{\operatorname{tg} \delta_{cp\varphi} \lim_{\beta \to \infty} C}{1 - e^{-\vartheta} + \operatorname{tg} \delta_{cp\varphi} \lim_{\beta \to \infty} C} . \qquad (15)$$

This equation enables us to choose the n needed to give a specified $\delta_m\phi$ in terms of the parameters of the resonant circuit, or vice versa. Little is gained from increasing the pulse length unduly. On the other hand, self-bias implies that the pulse length needed to give the specified $\delta_m\phi$ increases when a certain point β_{opt} has been passed. The cause of β_{opt} is that the cut-off angle starts to decrease rapidly as the external signal is increased in amplitude. An explicit expression for β_{opt} cannot be obtained, so it is best to find that quantity directly from (14).

The forms of the phasing function and of the statistical distribution agree well with those found by means of numerical solution of the equations for a related problem [1].

Figure 3 shows the time taken for an external signal to pull an oscillator with self-bias into phase as a function of the amplitude of that signal (here the phase is that of the two oscillations jointly, not the phase of the oscillator after the pulse has ended). The broken lines show the curves found

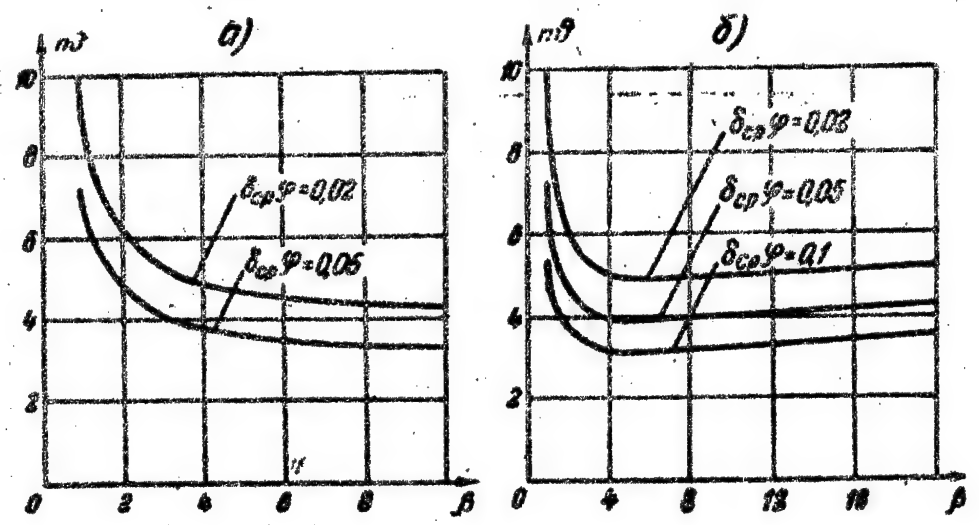

Fig. 2. Pulse length needed to give a specified $\delta_m\phi$
a) fixed bias (sin $\theta_0 = 1$, $e_0 = 0$),
b) self-bias (sin $\theta_0 = 1$, $\theta_g = 0$)

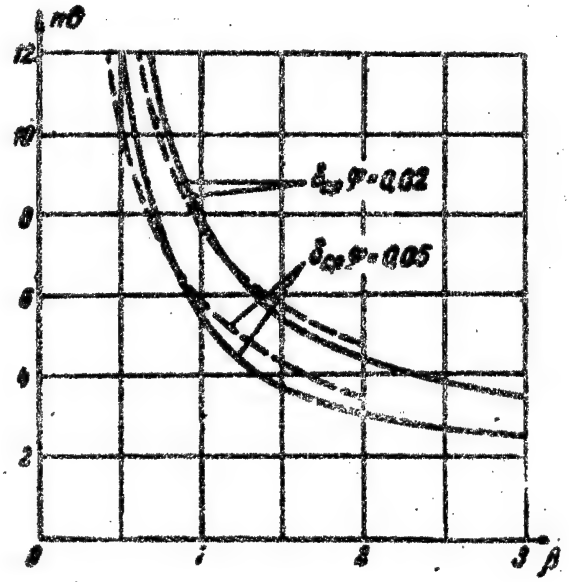

Fig. 3. Time taken for an external signal to pull an oscillator with self-bias into phase (sin $\theta_0 = 1$, $\theta_g = 0$)

from a numerical integration of the simplified equations. It is clear that the two sets of results agree quite well.

3. Small Degrees of Mistuning.

Here we use system (6) with the new variables of (7). The case of mistuning ($b \neq 0$) is treated as before; (6) is transformed to a set of finite-difference equations

$$\begin{aligned} \zeta_n &= \alpha_1 \zeta_{n-1} + \gamma_1 \eta_{n-1} + d_1; \\ \eta_n &= \alpha_1 \eta_{n-1} - \gamma_1 \zeta_{n-1} + l_1 \end{aligned} \tag{16}$$

the initial conditions being $\zeta_0 = \sin\varphi_0$, $\eta_0 = \cos\varphi_0$, , in which

$$\alpha_1 = B\cos b\vartheta; \quad \gamma_1 = B\sin b\vartheta;$$

$$d_1 = -C\sin(b\vartheta/4); \quad l_1 = C\cos(b\vartheta/4).$$

The solution to (16) is

$$\begin{aligned} \zeta_n &= \alpha_n \zeta_0 + \gamma_n \eta_0 + d_n; \\ \eta_n &= \alpha_n \eta_0 - \gamma_n \zeta_0 + l_n, \end{aligned} \tag{17}$$

in which α_n, γ_n, d_n, and i_n are given by the recurrence relations

$$\begin{aligned} \alpha_n &= \alpha_{n-1}\alpha_1 - \gamma_{n-1}\gamma_1; \\ \gamma_n &= \alpha_{n-1}\gamma_1 + \gamma_{n-1}\alpha_1; \\ d_n' &= \alpha_{n-1} d_1 + \gamma_{n-1} l_1 + d_{n-1}; \\ l_n &= \alpha_{n-1} l_1 - \gamma_{n-1} d_1 + l_{n-1} \end{aligned} \tag{18}$$

subject to the condition that α_1, d_1, γ_1, and l_1 are defined by (16). Then (16) with (18) gives us the coefficients as

$$\alpha_n = B^n \cos nb\vartheta; \quad \gamma_n = B^n \sin nb\vartheta;$$

$$d_n = C\sum_{k=0}^{n-1} B^k \sin(kb\vartheta - b\vartheta/4); \quad l_n = C\sum_{k=0}^{n-1} B^k \cos(kb\vartheta - b\vartheta/4).$$

Again, (17) gives us the phasing funotion as

$$\operatorname{tg}(\varphi_n - b\vartheta) = \frac{B^n \sin[\varphi_0 + (n-1)b\vartheta] + C\sum_{k=0}^{n-1} B^k \sin(kb\vartheta - b\vartheta/4)}{B^n \cos[\varphi_0 + (n-1)b\vartheta] + C\sum_{k=0}^{n-1} B^k \cos(kb\vartheta - b\vartheta/4)}. \quad (19)$$

Figure 4a shows that function; it is clear that there is only one point of stable equilibrium for any pulse length. The point of unstable equilibrium occuring for small pulse lengths vanishes when the length is made greater than the n defined by

$$B^n = C\sum_{k=0}^{n-1} B^k \cos(kb\vartheta - b\vartheta/4).$$

The statistical-distribution function for the phase is given by (19) as

$$W(\varphi_n) = \frac{\sqrt{B^{2n} - |K(\varphi_n, n)|^2} + \cos(\varphi_n - b\vartheta) C\sum_{k=0}^{n-1} B^k \cos(kb\vartheta - b\vartheta/4)}{2\pi\sqrt{B^{2n} - |K(\varphi_n, n)|^2}},$$

in which

$$K(\varphi_n, n) = C\cos(\varphi_n - b\vartheta)\sum_{k=0}^{n-1} B^k \sin(kb\vartheta - b\vartheta/4) -$$
$$-C\sin(\varphi_n - b\vartheta)\sum_{k=0}^{n-1} B^k \cos(kb\vartheta - b\vartheta/4).$$

Figure 4b illustrates the form of the function $W(\phi_n)$.

Figure 4c shows the phase after the pulse as a function of pulse length for several different initial phases. The phase change (relative to the steady-state phase) shows an oscillatory variation with the length of the pulse.

It is not possible to obtain a reasonably simple expression for the pulse length needed to produce a specified

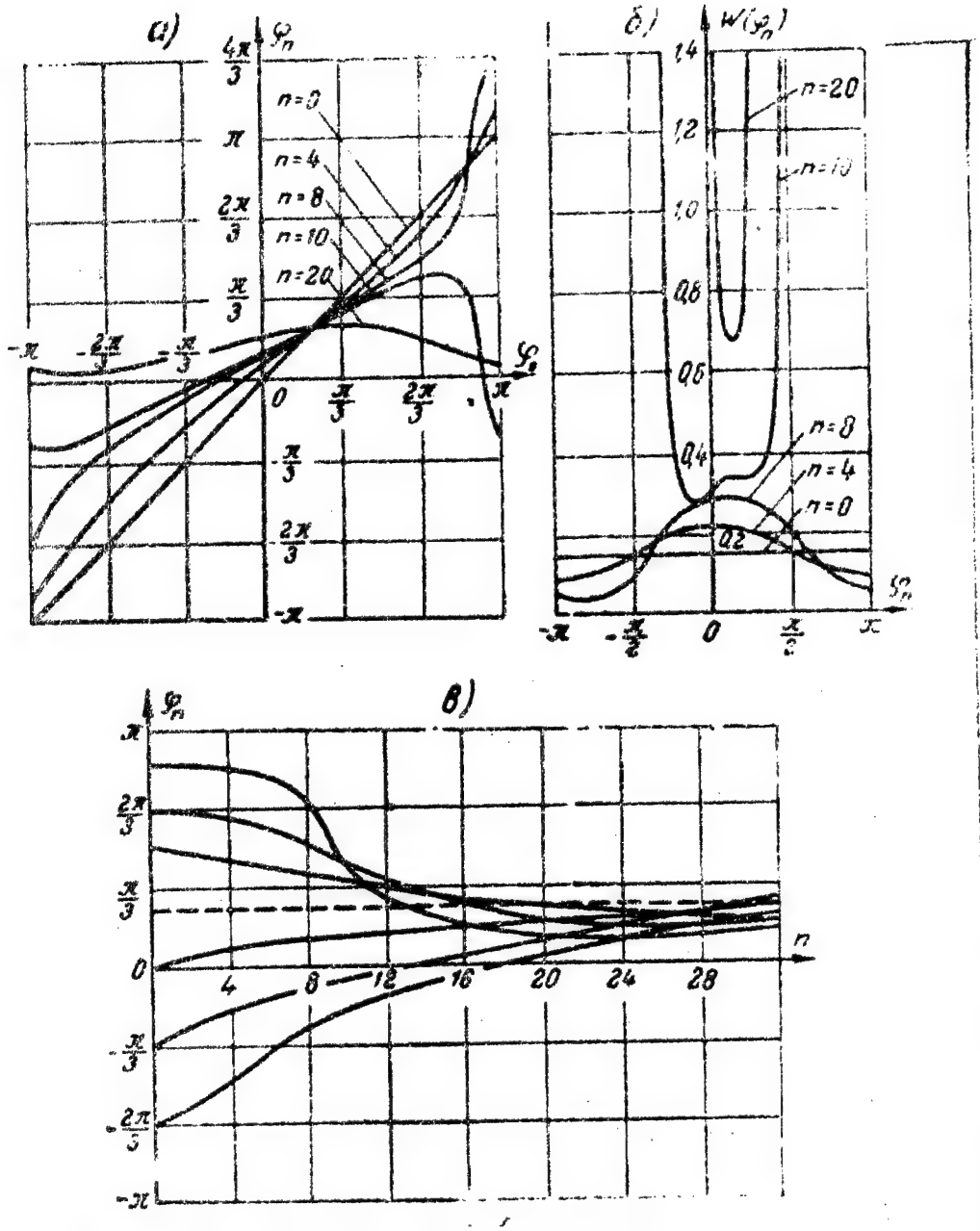

Fig. 4. Mistuning ($h=0,5$; $\beta=2$; $\sin\theta_0=1$; $\Delta=0,1$; $e_0=0$):
a) Phasing function, b) statistical-distribution function for the phase, e) phase when pulse has ended.

approach to the steadystate phase, but the approach itself may be found readily if the range of possible phases is found from the condition $W(\phi_n) \to \infty$. It is clear also that putting $b = 0$ causes (19) and (20) to become the expressions given above for the previous case.

The expression for the phasing function, (19) is the most complete solution of the problem.

The results from a numerical solution of the problem agree well with our expressions for the case of mistuning.

An analytic solution has been obtained in a reasonably complete form for the case for a radiofrequency pulse of large amplitude ($\beta > 1$) and of rectangular envelope. The method is suitable for use in considering similar pulses of small amplitude ($\beta < 1$); in that case we may neglect the change in the amplitude of the oscillation and need consider only the finite-difference equation for the phase. The results obtained in that way agree well with those given by others.

The phasing-function method can be used to consider many problems connected with nonlinear systems acted on by external forces, particularly problems in which other methods are not applicable, e.g. on account of the large amplitude of the force.

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THE SUFFICIENT CONDITIONS FOR DYNAMIC STABILITY IN A LARGE POWER GRID SUBJECT TO ALTERNATING ELECTROMOTIVE FORCES

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A particular controlled power grid is considered in order to demonstrate a new method of considering stability in general for simple types of power grid. The method is one in which the order of the nonlinear differential equations for the system is reduced (due allowance being made for the behavior of the system) by utilizing the fact that the various time-constants differ greatly in magnitude; Lyapunov's direct method is then used.

The stability of any particular part of power grid may be taken to be reasonably well understood, but the stability of the system as a whole has not been examined in any great detail with proper allowance for processes in the pipelines of hydroelectric stations, in the condensation plant of thermal power stations, in the control gear, in the generators, etc. The main papers [1-3] deal principally with the steady-state stability of simple systems (with stability in the restricted sense), although even that problem is very difficult on account of the large number of degrees of freedom. The dynamic (general) stability has hardly been considered theoretically at all, so far as I am aware. The practical problems are solved by means of numerical methods or by means of analogs and high-speed digital computers; the problem is reviewed in [4,5].

Here a particular controlled power grid is considered in order to demonstrate a new method of considering stability in general for simple types of power grid. The method is one in

which the order of the nonlinear differential equations for the system is reduced (due allowance being made for the behavior of the system) by utilizing the fact that the various time-constants differ greatly in magnitude; Lyapunov's direct method is then used. I have used this approach [6,7] to examine the dynamic stability of a controlled grid containing two hydroelectric stations not having derivative control (the electromagnetic transients in the generators were neglected). Here I derive the sufficient conditions for dynamic stability in a large power grid subject to alternating electromotive forces in order to illustrate the power of this approach more fully.

Consider a system containing two hydroelectric stations not having derivative control (or one such station and a thermal power station having a very short steam pipe), the two stations working in parallel into a common grid. I neglect fluctuations as between items of plant in the stations individually and replace each station by a single equivalent synchronous generator. I assume that the two equivalent generators are fitted with speed controls (ones with direct feedback) and that the second generator is fitted also with a voltage control. Further, I assume that the voltage supplied by the first station is kept constant at all times, that the second generator is of salient-pole type, and that various simplifying assumptions are justified (namely, that there are no damping windings, that the resistance of the stator circuit is negligible, that saturation is negligible, etc). Then the differential equations for the system take the form

equation of motion of the rotors:

$$T_{ai}\, ds_i / dt = M_i - P_i \qquad (i = 1,2), \tag{1}$$

in which s_i is the slip relative to a synchronously rotating field, M_i is the mechanical torque, P_i is the electrical torque, and T_{ai} is the time-constant of the machine; equation relating the relative angular divergence of the rotors to the slip:

$$T_0 d\delta_{12}/dt = s_1 - s_2, \tag{2}$$

in which T_0 is the time-constant;

equations describing the electromagnetic transient in the field circuits of the generator fitted with voltage control:

$$T_{do2}\, dE'_{d2}/dt + E_{d2} = E_{do2}; \tag{3}$$

$$T_{e2}\, dE_{de2}/dt + E_{de2} = E_{df2}; \tag{4}$$

$$T_{f2}\, dE_{df2}/dt + E_{df2} = E_{reg}, \tag{5}$$

in which E_{do2} is the steady-state no-load e.m.f., E_{d2} is the no-load e.m.f. in the transient state, E'_{d2} is the e.m.f. resulting from the transient reactance, E_{df2} is the e.m.f. corresponding to the voltage at the terminals of the exciter, E_{reg} is the e.m.f. corresponding to the voltage at the field terminals produced by the voltage control, and T_{do2}, T_{e2}, and T_{f2} are time-constants relating respectively to the field circuit with the stator circuit open, to the circuit of the exciter, and to the field circuit of the exciter;

equation for the speed control (mass of controller etc neglected):

$$T_{si}\, d\mu_i/dt = f_i(s_i - \rho_i \mu_i) \qquad (i = 1,2), \tag{6}$$

in which μ_i is the relative displacement of the servo's piston, and T_{si} is the time-constant for the servo.

Then (see [8], also [6, 7])

$$\dot{M}_i = \frac{\xi_{i0} - a_i \varphi_i^2}{1 - a_i}(1 - \tau_{ii} + c_i) - c_i \varphi_i^2 \qquad (i = 1,2), \tag{7}$$

$\varphi_i = \varphi_{i0} + s_i$, $\tau_{ii} = \tau_{ii0} + p_i$, ξ_{i0}, a_i, c_i are time-constants. I take the P_i (neglecting electrical damping) to be

$$P_1 = \frac{1}{\lambda_1}\left[\frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1 E_Q}{Z_{12}} \sin(\delta_{12} - \alpha_{12})\right];$$
$$P_2 = \frac{1}{\lambda_2}\left[\frac{E_2^2}{Z_{22}} \sin \alpha_{22} - \frac{E_1 E_Q}{Z_{12}} \sin(\delta_{12} + \alpha_{12})\right], \tag{8}$$

in which λ_i is the ratio of the nominal power of the i-th equivalent generator to the actual power, E_q is the e.m.f. of a fictitious machine not having salient poles but having equal quadrature and direct reactances, the Z_{ik} are self and mutual impedances (for which purpose the impedances are taken as equal to the synchronous quadrature reactances) and the α_{ik} are constant angles*.

The following finite relationships exist between the above variables [9]:

$$\frac{\lambda_2 E'_{d2}}{X_{q2} - X'_{d2}} = E_Q\left(\frac{\lambda_2}{X_{q2} - X'_{d2}} - \frac{\cos \alpha_{22}}{Z_{22}}\right) + \frac{E_1}{Z_{12}} \cos(\delta_{12} + \alpha_{12}); \tag{9}$$

$$E_{d2} = E_Q \frac{X_{d2} - X'_{d2}}{X_{q2} - X'_{d2}} - E'_{d2} \frac{X_{d2} - X_{q2}}{X_{q2} - X'_{d2}}; \tag{10}$$

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* The electrical torque M_e in dimensional variables is $w_e/\omega \approx w_e/\omega_0$, in which w_e is the electromagnetic power; the base power is w_b, so $M_e = (w_b/\omega_0)(w_e/w_b)$, and

$$P_1 = \frac{M_{e1}}{M_{1\,max}} = \frac{w_{\delta a3}}{\omega_0 M_{1\,max}} \frac{w_{e1}}{w_{\delta a3}} = \frac{1}{\lambda_1}\left[\frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_Q E_1}{Z_{12}} \sin(\delta_{12} - \alpha_{12})\right]$$

$$\lambda_2 U_{d2} = \lambda_2 (E'_{d2} - I_{d2} X'_{d2}) = \frac{E_1 X_{q2}}{Z_{12}} \cos(\delta_{12} + \alpha_{12}) + E_Q \left(\lambda_2 - \frac{X_{q2}}{Z_{22}} \cos \alpha_{22} \right); \quad (11)$$

$$E_{per} = E_{02} - k (U_2 - U_{20}); \quad (12)$$

$$U_{d2} = U_2 \cos \delta_2; \quad (13)$$

$$\operatorname{tg} \delta_2 = \frac{P_2 X_{q2}}{E_Q U_{d2}} \quad (14)$$

(see [10], p. 106). Here X_q is the quadrature synchronous reactance of the generator, X_d is the direct synchronous reactance, X'_d is the transient direct reactance, U_2 is the voltage at the output terminals, U_{d2} is the direct component of that voltage, and P_2 is defined by (8).

Equations (1)-(6) and (7)-(14) describe the system, but they contain 14 variables, namely φ_i, η_i, E_{dc2}, E_{d2}, E'_{d2}, E_{df}, E_{per}, δ_{12}, E_Q, U_{d2}, U_2, δ_2 $(i = 1,2)$. Those variables may be replaced by the deviations from the steady state, namely

$$\varphi_i = \varphi_{i0} + s_i; \qquad E_{per} = (E_{per})_0 + \varepsilon_{per};$$

$$\eta_i = \eta_{i0} + \mu_i; \qquad \delta_{12} = \delta_{120} + \Theta_{12};$$

$$E_{dc2} = (E_{dc2})_0 + \varepsilon_{dc}; \qquad E_Q = (E_Q)_0 + \varepsilon_Q;$$

$$E_{d2} = (E_{d2})_0 + \varepsilon_d; \qquad U_{d2} = (U_{d2})_0 + u_d;$$

$$E'_{d2} = (E'_{d2})_0 + \varepsilon'_d; \qquad U_2 = (U_2)_0 + u;$$

$$E_{df2} = (E_{df2})_0 + \varepsilon_{df}; \qquad \delta_2 = \delta_{20} + \Theta.$$

The steady-state quantities are defined by

$$\frac{\xi_{10} - a_1 \varphi_{10}^2}{1 - a_1} (1 - \eta_{10} + c_1) - c_1 \varphi_{10}^2 = \frac{1}{\lambda_1} \left[\frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1 (E_Q)_0}{Z_{12}} \sin(\delta_{120} - \alpha_{12}) \right];$$

$$\frac{\xi_{20} - a_2 \varphi_{20}^2}{1 - a_2} (1 - \eta_{20} + c_2) - c_2 \varphi_{20}^2 = \frac{1}{\lambda_2} \left[\frac{(E_Q)_0^2}{Z_{22}} \sin \alpha_{22} - \frac{E_1 (E_Q)_0}{Z_{12}} \sin(\delta_{120} + \alpha_{12}) \right]; \quad (15)$$

$$\varphi_{10} = \varphi_{20}; \tag{16}$$

$$(E_{d2})_0 = (E_{de2})_0 = (E_{df2})_0 = (E_{per})_0; \tag{17}$$

$$\frac{\varphi_{i0} - \varphi_{inom}}{\eta_{i0} - \eta_{inom}} = p_i \qquad (i = 1,2); \tag{18}$$

$$\frac{\lambda_2 (E'_{d2})_0}{X_{q2} - X'_{d2}} = (E_Q)_0 \left(\frac{\lambda_2}{X_{q2} - X'_{d2}} - \frac{\cos \alpha_{22}}{Z_{22}} \right) + \frac{E_1}{Z_{12}} \cos (\delta_{120} + \alpha_{12}); \tag{19}$$

$$(E_{d2})_0 = (E_Q)_0 \frac{X_{d2} - X'_{d2}}{X_{q2} - X'_{d2}} - (E'_{d2})_0 \frac{X_{d2} - X_{q2}}{X_{q2} - X'_{d2}}; \tag{20}$$

$$\lambda_2 (U_{d2})_0 = \frac{E_1 X_{q2}}{Z_{12}} \cos (\delta_{120} + \alpha_{12}) + (E_Q)_0 \left(\lambda_2 - \frac{X_{q2}}{Z_{22}} \cos \alpha_{22} \right); \tag{21}$$

$$(E_{per})_0 = E_{0\Sigma}; \tag{22}$$

$$(U_{d2})_0 = (U_2)_0 \cos \delta_{20}; \tag{23}$$

$$\operatorname{tg} \delta_{20} = (P_2)_0 X'_{q2} / (E_Q)_0 (U_{d2})_0. \tag{24}$$

(24)

Then (15-(24) can be used to represent (1)-(14) in terms of perturbations, namely as

$$T_{a1} \frac{ds_1}{dt} = - \frac{a_1 (2\varphi_{10} s_1 + s_1^2)}{1 - a_1} (1 - \eta_{10} + c_1 - p_1) - c_1 (2\varphi_{10} s_1 + s_1^2) -$$

$$- \frac{\xi_{10} - a_1 \varphi_{10}^2}{1 - a_1} p_1 - \frac{1}{\lambda_1} \left\{ \frac{E_1 |(E_Q)_0 + e_Q|}{Z_{12}} \sin (\delta_{120} + \Theta_{12} - \alpha_{12}) - \right.$$

$$\left. - \frac{E_1 (E_Q)_0}{Z_{12}} \sin (\delta_{120} - \alpha_{12}) \right\};$$

$$T_{a2}\frac{ds_2}{dt}=\frac{-a_2(2\varphi_{20}s_2+s_2^2)}{1-a_2}(1-\eta_{20}+c_2-\mu_2)-c_2(2\varphi_{20}s_2+s_2^2)$$

$$-\frac{\xi_{20}-a_2\varphi_{20}^2}{1-a_2}\mu_2-\frac{1}{\lambda_2}\left\{\frac{|(E_Q)_0+\varepsilon_Q|^2}{Z_{22}}\sin\alpha_{22}-\right. \tag{1a}$$

$$-\frac{E_1|(E_Q)_0+\varepsilon_Q|}{Z_{12}}\sin(\delta_{120}+\Theta_{12}+\alpha_{12})-\left[\frac{(E_1)_0^2}{Z_{22}}\sin\alpha_{22}-\right.$$

$$\left.\left.-\frac{E_1(E_Q)_0}{Z_{12}}\sin(\delta_{12}+\alpha_{12})\right]\right\};$$

$$T_0\frac{d\Omega_{12}}{dt}=s_1-s_2; \tag{2a}$$

$$T_{02}\frac{de'_d}{dt}+e_d=e_{de}; \tag{3a}$$

$$T_{e2}\frac{d\varepsilon_{de}}{dt}+\varepsilon_{de}=\varepsilon_{df}; \tag{4a}$$

$$T_{f2}\frac{d\varepsilon_{df}}{dt}+\varepsilon_{df}=\varepsilon_{per}; \tag{5a}$$

$$T_{s1}\frac{d\mu_1}{dt}=f_1(s_1-\rho_1\mu_1); \tag{6a}$$

$$T_{s2}\frac{d\mu_2}{dt}=f_2(s_2-\rho_2\mu_2);$$

$$\frac{\lambda_2 e'_d}{X_{q2}-X'_{d2}}=\varepsilon_Q\left(\frac{\lambda_2}{X_{q2}-X'_{d2}}-\frac{\cos\alpha_{22}}{Z_{22}}\right)+ \tag{9a}$$

$$+\frac{E_1}{Z_{12}}|\cos(\delta_{120}+\Theta_{12}+\alpha_{12})-\cos(\delta_{120}+\alpha_{12})|;$$

$$e_d=\varepsilon_Q\frac{X_{d2}-X'_{d2}}{X_{q2}-X'_{d2}}-\varepsilon'_d\frac{X_{d2}-X_{q2}}{X_{q2}-X'_{d2}}; \tag{10a}$$

$$\lambda_2 u_d=\varepsilon_Q\left(\lambda_2-\frac{X_{q2}}{Z_{22}}\cos\alpha_{22}\right)+ \tag{11a}$$

$$+\frac{E_1X_{q2}}{Z_{12}}|\cos(\delta_{12}+\Theta_{12}+\alpha_{12})-\cos(\delta_{12}+\alpha_{12})|;$$

$$\varepsilon_{per}=-ku; \tag{12a}$$

$$u_d=|(U_2)_0+u|\cos(\delta_{20}+\Theta)-(U_2)_0\cos\delta_{20}; \tag{13a}$$

$$\Theta=\operatorname{arctg}\frac{|(P_2)_0+P_2|X_{q2}}{|(E_Q)_0+\varepsilon_Q|\,|(U_{d2})_0+u_d|}-\operatorname{arctg}\frac{(P_2)_0X_{q2}}{(E_Q)_0(U_{d2})_0}. \tag{14a}$$

Figure 1 shows the block diagram corresponding to the problem when u_d, θ, and ε_{reg} have been eliminated. I assume that the equilibrium position corresponding to $s_1 = s_2 = \theta_{12} = \mu_1 = \mu_2 = \varepsilon_d' = \varepsilon_{de} = \varepsilon_{df} = 0$ is stable in the narrow sense, and set out to determine the extent of the region of stability in the phase space of the system.

I assume that the time-constants in (1a)-(6a) satisfy the conditions

$$T_0 \ll T_{s1},\ T_{e1},\ T_{f1} \ll T_{a1},\ T_{d02}, \tag{25}$$

which is usually the case (in a certain actual case the values were $T_0 = 0.00138$ sec, with T_{si}, T_{es}, and T_{f2} less than a tenth of a second, and T_{ai} and T_{d02} of the order of several seconds).

The problem then becomes simpler; (25) enables us to treat some of the equations as containing a small parameter in the derivatives and to divide the phase into regions of fast motion and regions of slow motion [11,12]. If we take the time-constants as being measures of the rates of change of the quantities, then the most rapidly varying one is θ_{12}, the next in order being μ_i, ε_{de}, and ε_{df}, followed by s_i and ε_d'. Now θ_{12} is an angle and appears in sine or cosine terms on the right, so the system may be taken as one having a rapidly changing phase as regards θ_{12} [13]. Such systems are treated by averaging in many cases, but we can see that any averaging with respect to θ_{12} on the right in (1a) - (14a) would be equivalent to splitting the system into two entirely independent stations, with the result that (in particular) we would be unable to establish the law followed by θ_{12} (because the equilibrium position of the averaged system would be displaced by a finite amount from the position for the initial system). Therefore I start not with θ_{12} but with μ_i, ε_{de}, and ε_{df},

whose magnitude decrease much more rapidly (any decrease in θ_{12} arises solely as a consequence of slow change in $s_1 - s_2$, as (2a) shows). Although those three variables change in much the same way, the problem is such that the behavior of μ_1 can be examined in isolation from the behavior of the other two. This feature is advantageous, in that the rate of change of μ_1 is independent of θ_{12}, whereas the rates of change of the other two are coupled (via ε_{reg}) to θ_{12}.

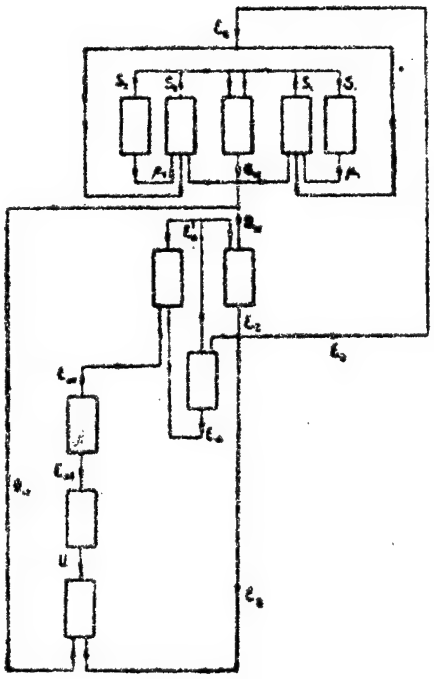

Fig. 1.

Let the functions f_1 and f_2 in (2a) (which specify the characteristics of the servomotors) satisfy the conditions

$$f_i(a_i)\,a_i > 0,\quad f_i(0) = 0 \qquad (26)$$

(σ_i being the argument, with i = 1,2) and be linearizable*. This implies that s_1 and s_2 vary much more slowly than μ_1 and μ_2, so we can put (6a) in the form

$$d\mu_1/dt = f_1(s_1 - \rho_1\mu_1), \qquad s_1 = \text{const};$$
$$d\mu_2/dt = f_2(s_2 - \rho_2\mu_2), \qquad s_2 = \text{const}. \tag{6b}$$

for the region in which μ_1 and μ_2 vary rapidly.

The points $f_1(s_1 - \rho_1\mu_1) = 0$ and $f_2(s_2 - \rho_2\mu_2) = 0$ correspond to states of equilibrium for the approximate equations (6b) for the fast motion. These states are stable if $\partial f_1/\partial\mu_1 < 0$ and $\partial f_2/\partial\mu_2 < 0$ at all points in the subspaces $f_1 = 0$ and $f_2 = 0$. This is always so if $\rho_1 > 0$, $\rho_2 > 0$, and (26) is satisfied. Therefore, if the image point is not near those subspaces at zero time, it is obliged to approach them as it moves along the fast loci and subsequently will move slowly within them. The restriction imposed on f_1 and f_2 are such that $f_i(\sigma_i) = 0$ only when $\sigma_i = 0$. Therefore the motion in the subspaces $f_1 = 0$ and $f_2 = 0$ is described by (1a)-(5a) and (9a)-(14a), in which $\mu_1 = s_1/\rho_1$ and $\mu_2 = s_2/\rho_2$, or

$$T_{a1}\frac{ds_1}{dt} = -C_1s_1 + D_1s_1^2 + H_1s_1^3 - \frac{E_1}{\lambda_1 z_{12}}\{|(E_Q)_0 + \varepsilon_Q|\sin(\delta_{120} + \Theta_{12} - \alpha_{12}) - (E_Q)_0\sin(\delta_{120} - \alpha_{12})\};$$

$$T_{a2}\frac{ds_2}{dt} = -C_2s_2 + D_2s_2^2 + H_2s_2^3 - \frac{1}{\lambda_2}\left\{|2(E_Q)_0\varepsilon_Q + \varepsilon_Q^2|\frac{\sin\alpha_{22}}{z_{22}} - \frac{E_1}{z_{12}}|((E_Q)_0 + \varepsilon_Q)\sin(\delta_{120} + \Theta_{12} + \alpha_{12}) - (E_Q)_0\sin(\delta_{120} + \alpha_{12})|\right\}. \tag{1a}$$

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* This argument can be used even if f_1 and f_2 are not linearizable provided that they satisfy (26).

Here

$$C_i = \frac{2a_i\varphi_{i0}}{1-a_i}(1-\eta_{i0}+c_i)+\frac{\xi_{i0}-a_i\varphi_{i0}^2}{(1-a_i)\rho_i}+2\varphi_{i0}c_i;$$

$$D_i = \frac{2a_i\varphi_{i0}}{(1-a_i)\rho_i}-\frac{a_i}{1-a_i}(1-\eta_{i0}+c_i)-c_i;$$

$$H_i = \frac{a_i}{(1-a_i)\rho_i} \qquad (i=1,2).$$

Equations (2a)-(5a) and (9a)-(14a) remain unchanged.

The order of the system is reduced from eight to sixth when μ_1 and μ_2 (the fast motions) are eliminated. I introduce new symbols $s_1 = x_1$, $s_2 = x_2$, $\theta_{12} = x_3$, $\varepsilon_Q = x_4$, $\varepsilon_{reg} = x_5$, $\varepsilon_{df} = x_6$, $T_{d1} = T_1$, $T_{d2} = T_2$, $T_0 = T_3$, $T_{reg0} = T_4$, $T_{r1} = T_5$, $T_{r2} = T_6$ and use (9a)-(14a) to eliminate ε_Q, ε_{reg}, ε_d, u_d, u and θ; then (1a)-(5a) become

$$T_1\frac{dx_1}{dt} = X_1(x_2, x_3, x_4); \quad T_2\frac{dx_2}{dt} = X_2(x_2, x_3, x_4);$$

$$T_3\frac{dx_3}{dt} = x_1 - x_2; \quad T_4\frac{dx_4}{dt} = X_4(x_3, x_4) + x_5; \qquad (27)$$

$$T_5\frac{dx_5}{dt} = x_6 - x_5; \quad T_6\frac{dx_6}{dt} = -x_6 + X_6(x_3, x_4).$$

I turn now to the behavior of x_5 and x_6. Let $T_1 z = T_3 x_3$; then the third equation in (27) implies that z varies with t at the same rate as do x_1, x_2, and x_4 (since $x_3 = T_1 z/T_3$ appears in sine and cosine terms in the right halves in (27)).

Then (25) enables us to assume that x_5 and x_6 vary much more rapidly than x_1, x_2, x_4, and z. The last two equations in (27) take the following forms for the regions in which x_5 and x_6 vary rapidly:

$$T_5 dx_5/dt = x_6 - x_5 \equiv F_1;$$
$$T_6 dx_6/dt = -x_6 + X_6(T_1 z/T_3,\ x_4) \equiv F_2 \quad (z = \text{const},\ x_4 = \text{const}). \tag{28}$$

The points in the subspace $F_1 = 0$, $F_2 = 0$ at which $x_5 = x_6 = X_6(T_1 z/T_3, x_4)$ are states of equilibrium for the fast motion described by (28). These states are stable, because the characteristic equation [12]

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_5} - T_5\lambda; & \frac{\partial F_1}{\partial x_6} \\ \frac{\partial F_2}{\partial x_5}; & \frac{\partial F_2}{\partial x_6} - T_6\lambda \end{vmatrix} = 0$$

has the negative real roots

$$\lambda_1 = -T_5, \qquad \lambda_2 = -T_6.$$

Therefore the image point moves along the loci for x_5 and x_6 in such a way as to approach the subspace $F_1 = F_2 = 0$, within which space it moves slowly in accordance with the first four equations in (27), in which $x_5 = X_6(x_3, x_4)$ (if we replace z by x_3). The loss of x_5 and x_6 reduces the system of equations to one of fourth order*. Lyapunov's direct method is used to examine the general stability of this nonlinear system, for which purpose we may use Aizerman's [14] approach or may take as our Lyapunov function V a quadratic sign-defined positive form and seek the

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* In fact, μ_1, μ_2, x_5, and x_6 tend simultaneously towards the states of equilibrium for the fast motions.

conditions that ensure that dV/dt (derived in accordance with the equations for the perturbed motion) is negative. Here we may transform the equations of motion to anonical form in advance, as in [7]. This gives us the sufficient conditions for stability in general to be imposed on the initial deviations.

Numerical Example

Let us suppose that hydroelectric station No. 1 has an F082 turbine whose reduced parameters are $a_1 = 0.079$ and $c_1 = 0.087$, and that No. 2 has an F-140 turbine whose parameters are $a_2 = -0.109$ and $c_2 = 0.0446$ (see [8], pp. 97-8). Both stations have no derivative control, so $\xi_i = \xi_{io} = 1$. Moreover, we are given the following parameters as

$E_{Q0} = 1,2$; $X_{q2} = 0,47$; $X_{d2} = 0,52$; $X'_{d2} = 0,2$; $Z_{11} = 0,8$; $\alpha_{11} = 18°40'$; $Z_{22} = 1$; $\alpha_{22} = 30°$; $Z_{12} = 3$; $T_{d1} = 10$; $T_{d2} = 12$; $T_0 = 0,00318$; $T_{d02} = 5$; $T_{e2} = 0,1$; $T_{f2} = 0,1$; $k = 0,145$. $\alpha_{12} = 5°$; $\lambda_1 = 1$; $\lambda_2 = 0,666$; $E_1 = 1$;

Here (15) gives us ϕ_{10} and δ_{120}; (18) gives the relation of η_{io} to ϕ_{io}, in which ζ_i is the residual unevenenness of control. I assume that $\varphi_{1\text{ном}} = \varphi_{2\text{ном}} = 1$; $\eta_{11\text{ном}} = 0,5$; $\eta_{12\text{ном}} = 0,4$; $\rho_1 = 0,05$; $\rho_2 = 0,06$. Then $\eta_{10} = (\varphi_{10} - 1)/\rho_1 + 0,5$; $\eta_{20} = (\varphi_{20} - 1)/\rho_2 + 0,4$. Substituting into (15), we find that the condition is satisfied if $\phi_{10} = \phi_{20} = 1$ and $\delta_{120} = 14.65°$. Further, (21), (23) and (24) give us that $(U_{d2})_o = 0.960$, $(U_2)_o = 0.985$, and $(\delta_{20}) = 13°$.

After this, some simple manipulations of the first four equations in (27) give us that

$$\frac{dx_1}{dt} = -2,028x_1 - 0,039x_3 - 0,009x_4 + X'_1(x_1, x_3, x_4),$$

in which

$$X'_1 \equiv 0,373x_1^2 + 0,172x_1^3 - 0,006(\cos x_3 - 1) - 0,039(\sin x_3 - x_3) + \\ + \varepsilon_Q |0,006(\cos x_3 - 1) + 0,033 \sin x_3|;$$

$$\varepsilon_Q = 1,541x_1 + 0,069 \sin x_3 - 0,205(\cos x_3 - 1);$$

$$\frac{dx_2}{dt} = -1,480x_2 + 0,038x_3 - 0,209x_4 + X'_2(x_2, x_3, x_4),$$

and

$$X_2'(x_2, x_3, x_4) = -0{,}302x_2^2 - 0{,}137x_2^3 + 0{,}210\,|0{,}127\,(\cos x_3 - 1) -$$
$$-0{,}045\,(\sin x_3 - x_3)| + 0{,}047\,(\sin x_3 - x_3) + 0{,}017\,(\cos x_3 - 1) +$$
$$+ \varepsilon_Q\,|0{,}014\,(\cos x_3 - 1) + 0{,}039 \sin x_3| - 0{,}063\,\varepsilon_Q^2;$$

$$\frac{dx_3}{dt} = 314{,}465\,(x_1 - x_2);$$

$$\frac{dx_4}{dt} = -0{,}016x_3 - 0{,}328x_4 + 0{,}2x_5 + X_4'(x_3),$$

and where

$$X_4'(x_3) = 0{,}046\,(\cos x_3 - 1) - 0{,}017\,(\sin x_3 - x_3);$$
$$x_5 = 0{,}003x_3 - 0{,}073x_4 + X_5'(x_3, x_4) \qquad (29)$$

(X_6' is found from (11a)-(14a)). Here $X_i'(x_j)$ denotes the nonlinear functions that when expanded in powers of x_j start not below the second-order terms. The equations for the first approximation are

$$\frac{dx_1}{dt} = -2{,}028x_1 - 0{,}039x_3 - 0{,}009x_4;$$
$$\frac{dx_2}{dt} = -1{,}480x_2 + 0{,}038x_3 - 0{,}209x_4; \qquad (30)$$
$$\frac{dx_3}{dt} = 314{,}465\,(x_1 - x_2);$$
$$\frac{dx_4}{dt} = -0{,}016x_3 - 0{,}342x_4.$$

It is easy to demonstrate that the coefficients of the corresponding characteristic equation satisfy the Rauth-Gurvits conditions.

We introduce the form

$$V = \frac{1}{2} \sum_{i,j=1}^{4} B_{ij} x_i x_j \tag{31}$$

and require that its total derivative (compiled in accordance with (30)) must satisfy the condition that

$$dV/dt = -(x_1^2 + x_2^2 + x_3^2 + x_4^2). \tag{32}$$

We substitute in (32) for dV/dt and equate the coefficients of like terms on the two sides to obtain a system of linear algebraic equations that defines the B_{ij} in (31). Then we have

$B_{11} = 2252{,}624$; $B_{22} = 2547{,}433$; $B_{34} = -1{,}373$;
$B_{12} = -2376{,}468$; $B_{23} = -11{,}989$; $B_{44} = 28{,}963$.
$B_{13} = 14{,}521$; $B_{24} = -43{,}418$;
$B_{14} = 18{,}771$; $B_{33} = 0{,}661$;

Now (30) corresponds to stability, so V (taken with the B_{ij}) will be positive [15]. The time derivative of this form, derived in accordance with (29), is

$$dV/dt = +\sum_{i=1}^{4} -x_i^2 + X_i' \partial V/\partial x_i . \tag{33}$$

We see that $dV/dt < 0$ if the x_i are small; the values at which $dV/dt = 0$ define the bounds to the stability region. If the system is to be stable in general, it is sufficient (but not necessary) for the initial perturbations to lie within the region so defined.

Practical calculations are best done as follows. We substitute into (33) for X_j' and $\partial V/\partial x_i$, and specify all the x_j except one (e.g. x_1); then we find the value of x_1 corresponding to $dV/dt = 0$. Then we alter the value assigned to one of the x_j; in this way we can compile the (x_1, x_j) section of the

stability region about the equilibrium state $x_1 = x_2 = x_3 = x_4 = 0$. Figure 2 shows the stability region found in the (x_1, x_2) section for $x_3 = 0.2$ and $x_4 = 0.1$ for the numerical example above.

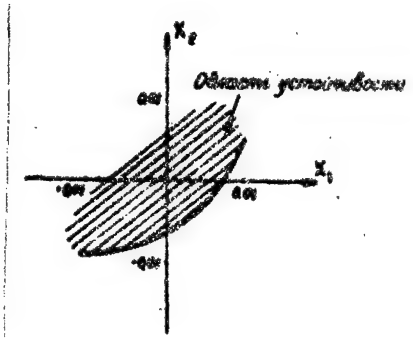

Fig. 2.

This method can be used to derive the sufficient conditions for stability in more complex circumstances (e.g. if both generators have voltage regulators). Digital computers are required to solve the equations if the system becomes very complex.

This problem was proposed by the All-Union Electric Power Research Institute (VNIIE).

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LINEAR DIGITAL FILTERS

Pages 494-505

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A method is presented for designing and evaluating linear algorithms used in approximate processes of differentiation, integration, and interpolation (or extrapolation) in digital computers; the frequency characteristic of an ideal operator is expanded in Fourier series form.

Control problems often involve algorithms that can be put in the form

$$\varphi(t+\tau)=\sum_{k=-s}^{0} a_k f(t+k\Delta t)+\sum_{k=1}^{n} b_k \varphi(t+\tau-k\Delta t), \qquad (1)$$

in which $\phi(t+\tau)$ is the result of applying the algorithm to a given function $f(t)$, τ is a constant for all t, a_k and b_k are constant coefficients defining the algorithm, s and n are positive integers, and Δt is the size of the steps in the independent variable used in $f(t)$ and $f(t+\tau)$.

Many approximate processes of differentiation, integration, and interpolation (or extrapolation) in digital computers use the formulas of the type of (1), e.g. Newton's formula [1].

The number of terms in (1) indicate the capacity of the store and the number of operations to be performed in order to use the algorithm.

Here I consider a special case in which (1) has no recurrent part; I shall show below that certain recurrent algorithms can be referred to this type.

1. Construction of the Algorithm

Any integro-differential operator, and any interpolation (extrapolation) operator can be described in terms of its Fourier transform (frequency characteristic).

For example

$$\frac{d}{dt} \doteq j\omega;$$

in general to any operator L of that form there corresponds a frequency character $F(\omega)$:

$$L \doteq F(\omega) = R(\omega) - jL(\omega).$$

If L transforms real functions to real ones, then $R(\omega)$ is an even function of ω, whereas $L(\omega)$ is odd.

Let us suppose that $R(\omega)$ and $L(\omega)$ can be expanded in Fourier-series form in the range $[-\Omega, +\Omega]$ (I shall show below that certain devices enable us to comply with this condition for very many of the operators encountered in practice). Then, when $-\Omega < \omega < \Omega$, we have

$$R(\omega) = \sum_{k=0}^{\infty} r_k \cos\left(k \frac{\pi}{\Omega} \omega\right);$$
$$L(\omega) = \sum_{k=0}^{\infty} l_k \sin\left(k \frac{\pi}{\Omega} \omega\right), \qquad (2)$$

in which the r_k and l_k are Fourier coefficients, and

$$F(\omega) = \sum_{k=0}^{\infty} r_k \cos\left(k \frac{\pi}{\Omega} \omega\right) - j \sum_{k=0}^{\infty} l_k \sin\left(k \frac{\pi}{\Omega} \omega\right). \qquad (3)$$

We restrict the numbers of terms in (2) to p + 1 and q + 1 respectively. Then

$$\tilde{F}(\omega) = \sum_{k=0}^{p} r_k \cos k \frac{\pi}{\Omega} \omega - j \sum_{k=0}^{q} l_k \sin k \frac{\pi}{\Omega} \omega \qquad (3a)$$

approximates (with a certain degree of error) to $F(\omega)$ in the range $[-\Omega, +\Omega]$; that error can be estimated as the standard deviation

$$\varepsilon = 2\Omega^{-1}\left[\int_0^{\Omega}\left[R(\omega)-\sum_{k=0}^{p} r_k \cos k\frac{\pi}{\Omega}\omega\right]^2 d\omega + \int_0^{\Omega}\left[L(\omega)-\sum_{k=0}^{q} l_k \sin k\frac{\pi}{\Omega}\omega\right]^2 d\omega\right]. \quad (4)$$

The reverse Fourier transform, as applied to the approximation $\tilde{F}(\omega)$, gives an approximate expression for the operator itself as

$$\tilde{L}(t)=\sum_{k=0}^{s}\frac{r_k+l_k}{2}\delta\left(t-k\frac{\pi}{\Omega}\right)+\sum_{k=0}^{s}\frac{r_k-l_k}{2}\delta\left(t+k\frac{\pi}{\Omega}\right), \quad (5)$$

in which $\delta(t)$ is a delta function, s is the larger of p and q, and the missing r_k or l_k are taken as zero.

Then the result of applying the approximate operator k to f(t) is put in the form

$$\varphi(t+\tau)=\int_{-\infty}^{+\infty}\tilde{L}(\tau)f(t-\tau)\,d\tau=\sum_{k=0}^{s}\frac{r_k+l_k}{2}f\left(t-k\frac{\pi}{\Omega}\right)+ \sum_{k=0}^{s}\frac{r_k-l_k}{2}f\left(t+k\frac{\pi}{\Omega}\right)=\sum_{k=-s}^{s}a_k f(t+k\Delta t), \quad (6a)$$

in which

$$\Delta t=\pi/\Omega, \qquad a_k=\begin{cases}(r_k-l_k)/2 & (k>0)\\ r_k & (k=0)\\ (r_k-l_k)/2 & (k<0)\end{cases} \quad (6b)$$

and, from (2),

$$r_k=\frac{2}{\Omega}\int_0^{\Omega}R(\omega)\cos k\Delta t\,\omega\,d\omega;\; l_k=\frac{2}{\Omega}\int_0^{\Omega}L(\omega)\sin k\Delta t\omega d\omega. \quad (6c)$$

Now s has been taken as the larger of p and q, but it is always convenient to put p = q = s, because the number of terms in (6a) depends only on s and because the ε given by (4) becomes small as p (or q) increases.

We take ε as our measure of the performance in the range $[-\Omega, +\Omega]$, and then we can select the coefficients in accordance with (6b) and (6c) in such a way as to minimize that error; again, we can compare different algorithms as regards the values they give for that error.

A definite physical meaning can be assigned to (4) if an energy spectrum $H^2(\omega)$ can be defined for f(t), or if we know at least the frequency range within which the major part of it falls. Then (4) gives us an estimate of the error when $H^2(\omega)$ = constant in the range $[0, \Omega]$ and $H^2(\omega) = 0$ outside that range. The general Fourier expansion which incorporates weighting for $H^2(\omega)$ must be used if $H^2(\omega)$ varies within the range $[0, \Omega]$.

In that case we have

$$\varepsilon = \frac{2}{\Omega}\left[\int_0^{\Omega} H^2(\omega)\left[R(\omega) - \sum_{k=0}^{s} r_k \cos k\Delta t\,\omega\right]^2 d\omega + \int_0^{\Omega} H^2(\omega)\left[L(\omega) - \sum_{k=0}^{s} l_k \sin k\Delta t\,\omega\right]^2 d\omega\right], \tag{7}$$

and the r_k and l_k are to be found from the two sets of linear equations derived from the conditions

$$\partial\varepsilon/\partial r_k = 0, \qquad \partial\varepsilon/\partial l_k = 0.$$

If we choose the r_k and l_k in this way, (7) then gives us the minimum error possible for that number of terms in the algorithm.

The set of equations for the r_k takes the form

$$\sum_{k=0}^{s} r_k \int_0^{\Omega} H^2(\omega) \cos k\Delta t\omega \cos i\Delta t\omega d\omega = \int_0^{\Omega} H^2(\omega) R(\omega) \cos i\Delta t\omega d\omega \qquad (8a)$$

$$(i = 0,1, \ldots, s).$$

The set for the l_k takes a similar form:

$$\sum_{k=0}^{s} l_k \int_0^{\Omega} H^2(\omega) \sin k\Delta t\omega \sin i\Delta t\omega d\omega = \int_0^{\Omega} H^2(\omega) L(\omega) \sin i\Delta t\omega d\omega \qquad (8b)$$

$$(i = 0, 1, \ldots, s).$$

We combine (8a) and (8b) term by term in accordance with (6b) to get a system of (2s + 1) equations for the coefficients in the algorithm:

$$\sum_{k=-s}^{s} a_k \int_0^{\Omega} H^2(\omega) \cos(k + i)\Delta t\omega d\omega = \int_0^{\Omega} H^2(\omega) |R(\omega) \cos i\Delta t\omega + L(\omega) \sin i\Delta t\omega| d\omega \qquad (8c)$$

$$(i = 0, +1, \ldots, +s).$$

If $\tau = 0$, (6a) is expressed by a formula of central type having (2s + 1) terms; it is a special case of (1). That algorithm provides the least error in approximating the real and imaginary parts of the frequency characteristic, each taken separately.

That formula can be used for any number of terms (formula (1)), the only difference being that only the δ of (7) will be minimal if the number of terms is even. Therefore in future (8c) or (1) will be used with the upper limit of the summation set as zero.

It is true that we have chosen a somewhat special case by taking a function having a bounded spectrum such that a suitable choice of Δt will make the error given by (7) an

almost complete estimate of the performance of the algorithm. A much wider class is represented by a function having a bounded energy (i.e. one for which $\int_0^\infty H^2(\omega)\, d\omega$ exists); here, in principle, one must allow for the additional error arising in the range $[\Omega, \infty]$.

The error needs careful consideration if (as above) f(t) has nearly a continuous spectrum. The method gives us a continuous analog to the algorithm (Fig. 1a), the analog being a linear four-terminal network consisting of a delay line and amplifiers. The corresponding digital analog has to be fitted at the input and output with pulse-forming units operating in synchronism at intervals Δt.

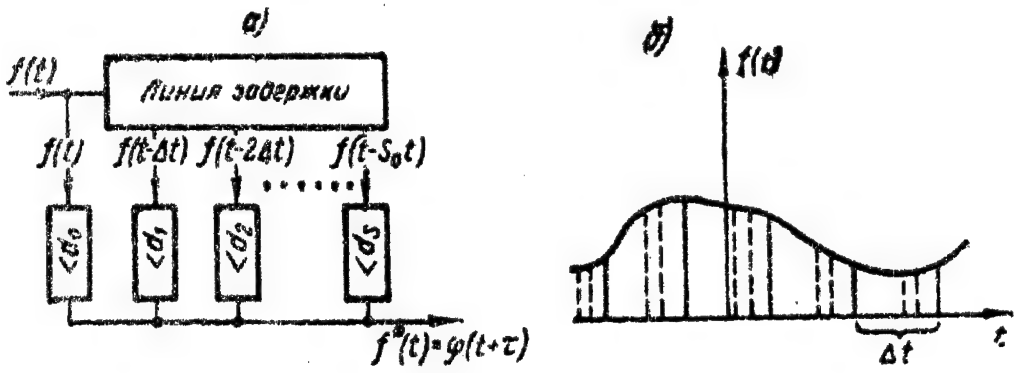

Fig. 1.

Now ϵ has been estimated without allowance for the quantization in time, so that error is the mean of all possible values for the time shift of the operation of thepulse devices acting on f(t) (Fig. 1b). That quantization is also a nonlinear operation, since it transforms the spectrum in the range $[\Omega, \infty]$ to the range $[0, \Omega]$, which causes further error. I shall not deal with these aspects of the matter.

2. Properties of the Algorithm for Approximate Interpolation or Extrapolation

The frequency characteristic of the interpolation operator is

$$F(\omega) = e^{j\omega\tau} = \cos\omega\tau + j\sin\omega\tau.$$

Then (8c) takes the form

$$\sum_{k=-s}^{0} a_k \int_0^{\Omega} H^2(\omega)\cos(k+i)\Delta t\omega d\omega = \int_0^{\Omega} H^2(\omega)\cos\omega(\tau + i\Delta t)\,d\omega \qquad (9)$$

$$(i = 0,1,2,\ldots,s).$$

The a_k given by (9) are functions of τ; used with (1) they can give an interpolation function $f_i(\tau)$ from the set of values of f(t):

$$f_n(\tau) = \sum_{k=-s}^{0} a_k(\tau) f(k\Delta t). \qquad (10)$$

Let us consider the main properties of this function as derived from the condition that the standard deviation is least in the range $[0, \Omega]$ (see section 1).

1. Firstly, $f_i(\tau)$ interpolates exactly between values of f(t), i.e. it takes the values of the latter at the points $\tau = k\Delta t$.

This can be demonstrated by substituting into (9) for each $\tau = n\Delta t$ (where n = -s,..., 0) the values $a_n = 1$, $a_k = 0$ (k = -s,..., n - 1, n + 1,..., 0).

2. Secondly, $f_i(\tau)$ can be represented in the form

$$f_k(\tau) = \sum_{k=s}^{0} A_k R(\tau + k\Delta t), \tag{11}$$

in which

$$R(\tau + k\Delta t) = \int_0^{\Omega} H^2(\omega) \cos\omega(\tau + k\Delta t)\, d\omega \Big/ \int_0^{\Omega} H^2(\omega)\, d\omega, \tag{12}$$

and the A_k are given by

$$A_k = \sum_{i=-s}^{0} b_{k,i} f(i\Delta t). \tag{13}$$

Omitting obvious steps, we see that the $b_{k,i}$ are the elements of a matrix that is the reciprocal to the matrix that is the reciprocal to the matrix for the coefficients of (8c).

Form (11) for the interpolation function is analogous to an interpolation polynomial in terms of a linear combination of Lagrange interpolation coefficients (if the points are equidistant), the only difference being that the linear combinations of (13) replace the specified values at the given points in (11).

Then $R(\tau + k\Delta t)$ may be called the generalized Lagrange interpolation coefficient; physically it is the normalised autocorrelation function corresponding to a spectrum equalling $H^2(\omega)$ within the range $[0, \Omega]$ and zero outside that range.

3. Thirdly, $f_1(\tau)$ at the limit (when $\Omega_1 \to 0$) goes over to a Lagrange interpolation polynomial differing from zero only in the range $[0, \Omega_1]$ and being continuous within that range (provided that $\Delta t = \pi/\Omega$ = constant).

Let us consider (9) again. First we divide the

i-th equation by $\int_0^{\Omega} H^2(\omega)\, d\omega$ (the normalizing factor) to get

$$\sum_{k=-s}^{0} a_k R\left[(k+i)\Delta t\right] = R(\tau + i\Delta t) \quad (i = 0, 1, 2, \ldots, s). \tag{14}$$

We change the integration limit in (12) from Ω to Ω_1 and pass to the limit $\Omega_1 = 0$; then (14) becomes

$$\lim_{\Omega_1 \to 0} \sum_{k=-s}^{0} a_k(\tau) = 1. \tag{15}$$

But (15) is quite inadequate to define the a_k, so we compile a new system by subtracting the (i + 1)th equation from the i-th in (14):

$$\sum_{k=-s}^{0} a_k \left\{ R\left[(k+i)\Delta t\right] - R\left[(k+i+1)\Delta t\right] \right\} = R(\tau + i\Delta t) - R\left[\tau + (i+1)\Delta t\right],$$

$$(i = 1, 2, \ldots, s). \tag{16}$$

The various coefficients tend to zero at different rates as $\Omega_1 \to 0$. We devide each equation in (16) by normalizing factor for a_{-s}, which gives us a system of equations whose coefficients do not tend to zero as $\Omega_1 \to 0$. In fact

$$\lim_{\Omega_1 \to 0} \frac{R(m\Delta t) - R\left[(m+1)\Delta t\right]}{R(n\Delta t) - R\left[(n+1)\Delta t\right]} = \frac{2m+1}{2n+1} \tag{17}$$

(we use the fact that $H^2(\omega)$ is continuous to find the value at the limit).

We combine the limit of (16) for $\Omega_1 \to 0$ with (15) to get our final system as

$$\sum_{k=-s}^{0} a_k (k+i)\Delta t = \tau + i\Delta t \qquad (i = 0, 1, 2, \ldots, s). \tag{18}$$

The a_k are clearly functions of τ.

Now I shall demonstrate that the Lagrange interpol-

ation coefficients form the solution to (18), i.e. that

$$a_k = L_k^s(\tau) = \frac{M_s(\tau)}{k!\,(s-k)!\,(\tau+k\Delta t)}(-1)^k \quad (k = 0, 1, 2, \ldots, s), \tag{19}$$

in which $M_s(\tau)$ is a factorial polynomial of degree s + 1:

$$M_s(\tau) = \tau(\tau+\Delta t)(\tau+2\Delta t)\ldots(\tau+s\Delta t). \tag{20}$$

For, consider any equation in (18). The a_k are the values taken by the quantity $\tau + i\,\Delta t$ on the right at the points $\tau = -s\,\Delta t$, $(-s+1)\,\Delta t, \ldots, 0$. Therefore the left part becomes a Lagrange interpolation polynomial when the substitution ismade from (19); that polynomial is constructed from the values of $\tau + i\,\Delta t$ at those same points.

Now, if $s \geqslant 1$, the interpolation polynomial becomes exactly the same as the one to be interpolated, so (19) is in fact the solution to (18); it is also the solution to (15), for the sum of the Lagrange factors equals one exactly.

So when $\Omega_1 \to 0$ we have

$$\lim_{\Omega_1 \to 0} f_n(\tau) = \sum_{k=-s}^{0} f(k\Delta t) \lim_{\Omega_1 \to 0} a_k = \sum_{k=-s}^{0} f(k\Delta t)\, L_{-k}^s(\tau). \tag{21}$$

Figure 2 shows the $f_i(\tau)$ for various values of Ω_i corresponding to $H^2(\omega)$ = constant in the range $[0, \Omega]$. We see that $f_i(\tau)$ becomes more and more like a parabola as Ω_1 decreases.

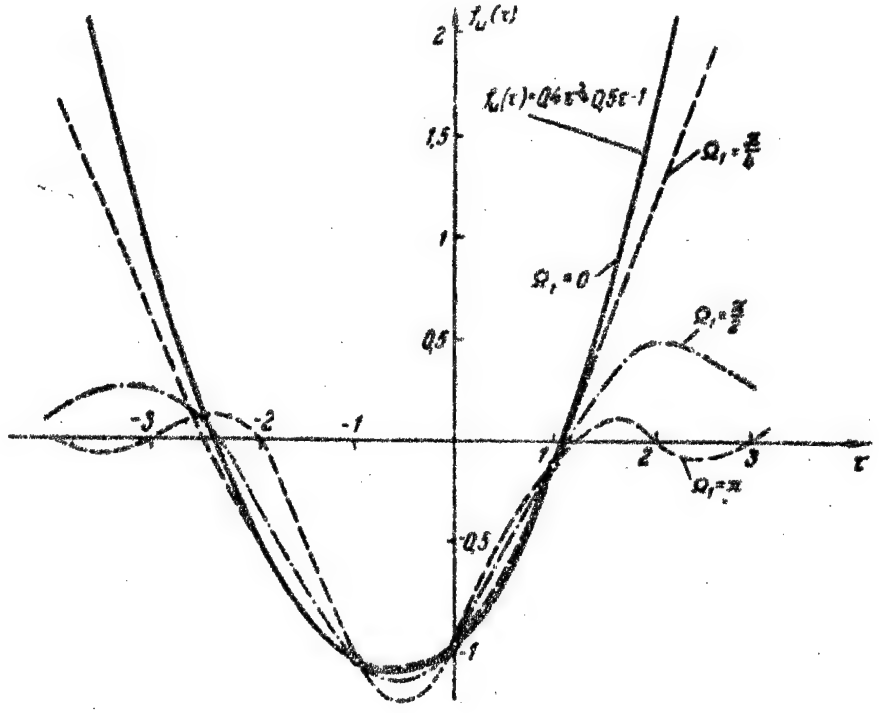

Fig. 2.

The passage to the limit in (21) may be considered as a passage to the spectrum $H^2(\omega) = \delta(\omega)$.

3. Algorithms for Approximate Differentiation and Integration

The coefficients of the algorithm may be found from (8c) or by means of (11); in the first case the frequency characteristic has to be expanded as a Fourier series, which is unsatisfactory for the integration operator

$$F_{инт}(\omega) = 1/j\omega . \qquad (22)$$

That difficulty may be overcome by choosing for the operator a frequency characteristic corresponding to an increment in the integral.

That characteristic for an increase s Δt in the independent variable (only intervals that are multiples of Δt are convenient) is

$$F_{\Delta\,int}(\omega) = \frac{1 - e^{-j\omega s\Delta t}}{j\omega} = \frac{\sin(\omega s\Delta t)}{\omega} + j\frac{\cos(\omega s\Delta t) - 1}{\omega}. \qquad (23)$$

We apply to (23) the method given in section one. In the second case the formulas are found directly by differentiating (integrating) (11), in the manner used to find formulas for approximate differentiation (integration) from the Lagrange interpolation polynomial.

The second method is exactly equivalent to the first. In fact, the frequency characteristic of (11) is

$$G(\omega) = \begin{cases} H^2(\omega)\sum\limits_{k=-s}^{0} A_k e^{j\omega k\Delta t} & (\omega = [0, \Omega]) \\ 0 & (\omega = [\Omega, \infty]) \end{cases}. \qquad (24)$$

The result of applying to $f_i(\tau)$ any integro-differential operator whose spectrum is $F(\omega)$ can be expressed (by virtue of (24)) by means of a reverse Fourier transform:

$$L[f_{in}(\tau)] = \int_{-\Omega}^{+\Omega} F(\omega) H^2(\omega) \left[\sum_{k=-s}^{0} A_k e^{j\omega k\Delta t}\right] e^{j\omega\tau} d\omega = \qquad (25)$$

$$= \sum_{k=-s}^{0} A_k \left\{\int_{-\Omega}^{+\Omega} H^2(\omega)\,[R(\omega)\cos\omega(\tau + k\Delta t) + L(\omega)\sin\omega(\tau + k\Delta t)]\,d\omega\right\}.$$

Again, (25) can be derived directly from (8c) by means of the device adopted to demonstrate property 2 (second section).

section).

In conclusion I give some examples of formulas for numerical differentiation and integration.

1. Numeral differentiation formula for $H^2(\omega) = 1/\pi$ for the range $\omega = [0,\pi]$ with $\Delta t = 1$.

Any of the above methods gives us the general relation

$$f'(\tau) \simeq -\sum_{k=-s}^{0} f_k \left\{ \frac{\sin|\pi(\tau-k)|}{\pi(\tau-k)^2} - \frac{\cos|\pi(\tau-k)|}{\tau-k} \right\}. \qquad (26)$$

A measure of the utility of the formula is given by the rate at which the moduli of the coefficients decrease, because few terms are needed if that rate is large. The choice $\tau = (2n+1)/2$, where $2n+1 = s$, gives the largest possible rate, for $\cos[\pi(\tau - k)]/(\tau - k)$ decreases roughly as $1/k$, so the coefficients decrease roughly as $1/k^2$. Then

$$f'\left(-\frac{s}{2}\right) = \frac{4}{\pi}\left[\frac{f\left(-\frac{s-1}{2}\right) - f\left(-\frac{s+1}{2}\right)}{1^2} - \frac{f\left(-\frac{s-3}{2}\right) - f\left(-\frac{s+3}{2}\right)}{3^2} + \cdots\right]. \qquad (27)$$

Figure 3a shows frequency characteristics and the (for the range $[0,\pi]$) given by (27) and by the usual polynomial formula with the same number of terms ($s = 5$) [1]:

$$f'\left(-\frac{5}{2}\right) = \frac{f(-2) - f(-3)}{128}\,150 - \frac{f(-1) - f(-4)}{3 \cdot 128}\,25 + \frac{f(0) - f(-5)}{5 \cdot 128}\,3. \qquad (28)$$

Figure 3b shows the distributions of the a_k functions of k for these formulas.

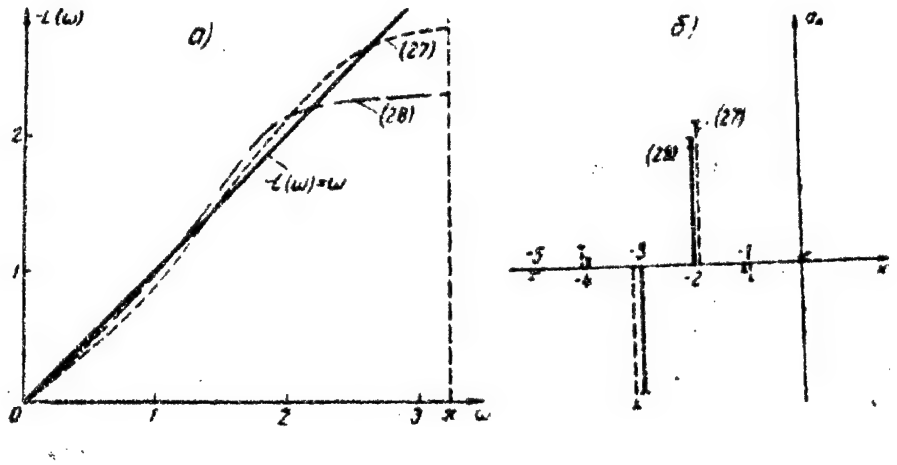

Fig. 3.

Each a_k indicates the extent to which the value of f(t) at the point $t - k\Delta t$ affects the current value (in this case, of the derivative); here t is the time after f(t) has been measured (determined).

2. Numeral integration formula for $H^2(\omega) = 1/\pi$ for the range $\omega = [0, \pi]$ with $\Delta t = 1$.

Here

$$\int_{t-s}^{t} f(t)\,dt = \sum_{k=-s}^{0} f(t+k)\frac{\mathrm{Si}\,[(s-k)\pi] + \mathrm{Si}\,(k\pi)}{\pi} \qquad (29)$$

Figure 4 shows the distributions of the a_k as functions of k; the broken line corresponds to the coefficients in Newton's formula. We see that the distribution is close to that for the simple trapezium rule.

This result is not unexpected, for the rapid increase in the error of interpolation with frequency is compensated to a large extent by the rapid fall in $1/\omega$, the frequency

characteristic of the integration operator.

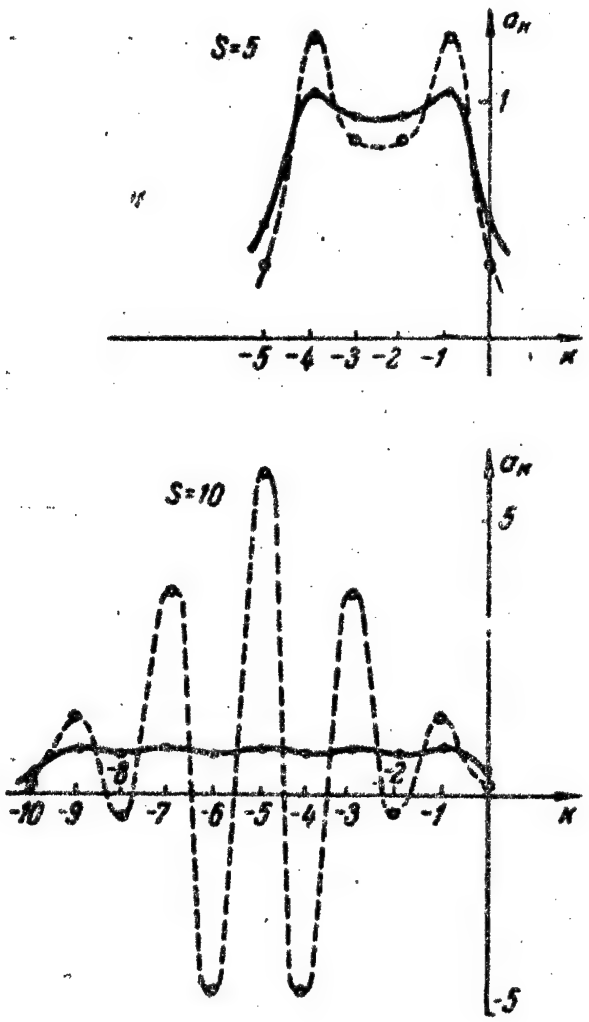

Fig. 4.

4. Remarks on the Algorithms for General Filters

My method enables one to construct an algorithm designed to reproduce for a digital computer the frequency response $F(\omega) = R(\omega) + jL(\omega)$ of a filter (provided that $R(\omega)$ is even and $L(\omega)$ odd). An extra requirement is often imposed on the algorithm, namely that the delay τ introduced by the algorithm must be minimal (e.g. if the computer forms part of an automatic control system).

In principle this method imposes no restriction on τ, which can be either positive or negative. However, in practice we find that ε increases rapidly if we try to make the delay zero or negative, so $\tau \geqslant 0$. For example, if the frequency characteristic satisfies the condition of physical feasibility.

$$\int_0^\infty R(\omega)\cos\omega\tau d\omega = \int_0^\infty L(\omega)\sin\omega\tau d\omega, \quad \tau > 0, \tag{30}$$

and if Ω is taken to be such

$$\int_\Omega^\infty |R(\omega)|\,d\omega \ll \int_0^\infty |R(\omega)|\,d\omega;$$

$$\int_\Omega^\infty |L(\omega)|\,d\omega \ll \int_0^\infty |L(\omega)|\,d\omega,$$

then (6a) gives us that the algorithm corresponding to zero delay implies small mean-square errors.

If, however, the characteristic does not satisfy (30), the error increases rapidly as we approach zero delay. The error is related to the parameter that specifies the degree of deviation from the condition, namely

$$\gamma = \frac{\int_0^{\infty} \left\{ \int_0^{\infty} [R(\omega)\cos(\omega\tau) - L(\omega)\sin(\omega\tau)]\, d\omega \right\}^2 d\tau}{\int_0^{\infty} [R^2(\omega) + L^2(\omega)]\, d\omega} \qquad (31)$$

(the two increase together).

Figure 5a shows the frequency characteristics of a single-stage RC filter; Figs. 5b and 5c show characteristics for low-frequency filters corresponding to different values of γ, and also the distribution of the a_k for $H^2(\omega) = 1/\pi$, $\Omega = \pi$. We see that neglect of the coefficients for $\tau > 0$ introduces substantial errors in the cases of Figs. 5b and 5c, because in those cases the mean-square error equals the sum of the squares of the omitted coefficients.

I have presented here a method of finding the coefficients in a linear algorithm designed to effect approximate interpolation, integration, or differentiation; the method is based on minimizing the mean-square error. It provides the best mean-square approximation to the Fourier transform of the interpolation or linear integro-differential operator within a specified frequency range $[0, \Omega]$. The mean-square error can be used to compare different approximate formulas. A similar method can be applied to algorithms designed to represent the frequency characteristic of a filter within a range $[0, \Delta]$.

The following results have been obtained in addition.

a) It has been shown that the interpolation function corresponding to pointwise interpolation between n equally spaced values of a function can be represented as a

linear combination of n generalized Lagrange interpolation coefficients $R(T + k\,\Delta t)$ (with $k = 1,\ldots n$), in which Δt is the spacing between values of the function. As generalized Lagrange coefficient we may take any function having the energy spectrum $H^2(\omega)$ within the bounded range $[0, \pi/\Delta t]$. This ensures that the meansquare error is minimal for the approximation to the frequency characteristic within that range for the spectrum $H^2(\omega)$. Parabolic interpolation is a special case, which corresponds to $H^2(\omega) = \delta(\omega)$.

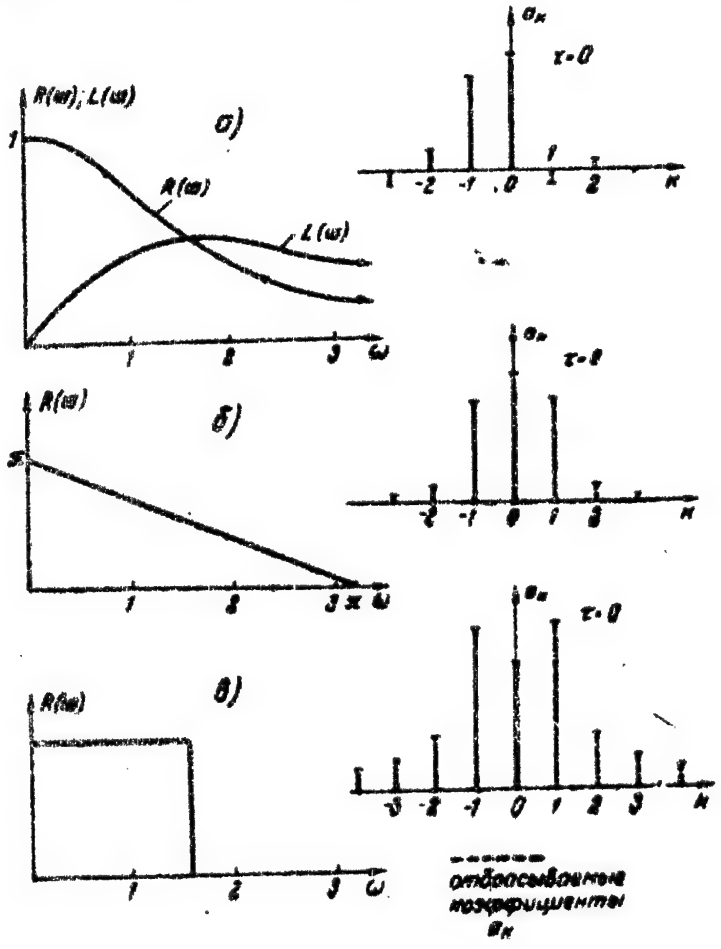

Fig. 5.

b) It is possible to derive formulas for approximate differentiation (integration) that satisfy the condition that the mean-square error must be minimal in the range $[0,\Omega]$ for a spectrum $H^2(\omega)$ by applying a general method or directly by differentiating (integrating) the corresponding interpolation function in a fashion analogous to that employed to derive formulas for numerical integration (differentiation) from an interpolation polynomial.

I wish to thank Yu. A. Shreider for his interest in the work and for valuable discussions.

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Submitted 25 April, 1959

SHORT COMMUNICATIONS AND LETTERS TO THE EDITOR.

LUNAR 1.63 cm RADIO EMISSION

Pages 506-507

by M.R. Zelinskaya, V.S. Troitskiy and L.N. Fedoseyev.

In 1956-57 we made measurements of the effective temperature of the central part of the lunar disk as a function of its phase. The results obtained are sufficiently well approximated by the expression:

$$T_A = 224° - 36° \cos(\Omega t - 40°) \qquad (1)$$

($\Omega t = 0$ corresponds to new moon), whereas the theoretical dependence [2] of the brightness radio temperature of the center of the moon on the phase has the form:

$$T_A = 204 - 133° (1 + 2\delta + 2\delta^2)^{-1/2} \cos(\Omega t - \xi), \qquad (2)$$

where $\delta = \beta/\varkappa$ is the ratio of the depth of penetration of the electromagnetic wave $1/\varkappa$ to the depth of penetration of the thermal wave $1/\beta$ (β is the attenuation of a thermal wave, $\varkappa$ the attenuation of an electromagnetic wave in the lunar rock, depending on the physical and chemical characteristics of the lunar surface),

$$\operatorname{tg}\xi = \delta/(1-\delta). \qquad (3)$$

Within the limits of accuracy of the measurements the magnitude of the constant component coincided with that calculated. The value of δ calculated on the basis of (1) and (2) proved equal to 2.3 ± 0.2; expression (3) then gives a value $\xi = 35°$, which coincides within the limits of error with the experimental value $\xi = 40° \pm 7°$, obtained by us.

The results obtained agree well with investigations made earlier at other wavelengths (see Table 1).

The results of observations on the wavelengths 1.25, 1.63 and 3.2 cm yield the interesting law

$$\delta/\lambda \simeq \text{const.} \qquad (4)$$

Table 1

λ (cm)	3,2 [3]	1,63	1,25 [3]	0,86 [4]	0,8 [4]	0,15 [5]
δ	4	2,3	1,5	1,6	2,3	0,08
δ/λ (cm^{-1})	1,25	1,4	1,2	1,9	2,9	0,5

It is known that this relationship is the result of a property of solid dielectrics of preserving the loss tangent approximately constant over almost the whole centimeter range [7], i.e.

$$\operatorname{tg}\Delta = 4\pi\sigma(\omega)/\varepsilon\omega \simeq \text{const}, \qquad (5)$$

where $\sigma(\omega)$ is the equivalent electric conductivity.

Using the value of δ obtained and a thermal conductivity $k = 2.5 \times 10^{-6}$, obtained from optical data [8], it is easy to find $\varkappa = 0.2\ cm^{-1}$ and $\sigma = 7.9 \times 10^{8}$ CGSU. This gives a loss angle for lunar rock of approximately 2° and a depth of radiowave penetration ($\lambda = 1.63$ cm) $1/\varkappa = 5$ cm for a depth of thermal wave penetration $1/\beta = 2.2$ cm. Compared with terrestrial rock this is a rather large value for the conductivity but is not out of the question for rocks with a large content of oxides of potassium, sodium and iron.

At wavelengths of 8.6 mm, 8 mm and 1.5 mm the law (4) is evidently somewhat upset. In the region around $\lambda = 8$ mm δ/λ has a quasi-resonance trend. If we concede the correctness of this result, then it is necessary either to allow the existence of a certain "transparency" (decrease in $\varkappa$) of the lunar rock in the 8 mm range, which would be difficult to account for, or to assume that the thermal conductivity of the surface layers of the lunar crust (from which 8 mm radiation mainly comes) is less than that at greater depths. This conclusion may also be drawn from the fact that the lag in radio-emission at a wave length of 1.63 cm compared with the thermal phase apparently proves to be somewhat greater than required by a single-layer model of the structure of the lunar crust and more nearly satisfies a two-layer model. However, inaccuracy in the measurement of the phase lag and significant discrepancies between values of δ/λ for close wavelengths (for 8 mm [4] and 8.6 mm [6], for example) testify to the relative inaccuracy of existing methods of measuring lunar radio emission, when it is a matter of choosing between a single and a double-layer model of the structure of its surface layers. For this purpose we need investigations with telescopes of greater resolving power in the millimeter and centimeter wavebands.

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Submitted: 18 Feb. 1959

INVESTIGATION OF THE STABILITY OF THE FIXED POINT OF TRANSFORMATION IN CRITICAL CASES.

Pages 507-508

by Yu.I. Neymark.

In this paper I establish criteria of the stability and instability of the fixed point of reflection of Euclidean space onto itself in so-called critical cases. Part of the results formulated below was obtained in another paper by the author / 1 / dealing with the bifurcation of fixed points.

1. Let the point reflection T_n of an n-dimensional Euclidean space R_n onto itself, mapping the point $M(x_1,x_2,\ldots,x_m,\ldots,x_n)$ onto the point $\bar{M}(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$, be represented in the form

$$\bar{x}=f(x,y); \qquad \bar{y}=y^*(x)+g(x,y), \tag{1}$$

where x is an m-dimensional vector, the components of which are the first m coordinates of the point M, y is an (n - m)-dimensional vector, the components of which are the succeeding n - m coordinates of the point M and

$$|g(x, y)| < q(x, y)|y - y^*(x)|. \tag{2}$$

We shall consider the reflection T_m of an m-dimensional space onto itself, defining it as follows:

$$\bar{x} = f[x, y^*(x)]. \tag{3}$$

It is evident that if x^* is a fixed point of the reflection T_m, then $[x^*, y^*(x^*)]$ will be a fixed point of the reflection T_n.

Theorem 1. If in the neighborhood of the fixed point x^* $V(x)$ is a Lyapunov function [2] for the reflection T_m and in a certain neighborhood of the fixed point $[x^*, y^*(x^*)]$ of the reflection T_n

$$\begin{gathered} |q(x, y)| < q < 1; \\ |V[f(x, y)] - V[f(x, y^*(x)]| < B|y - y^*(x)|, \end{gathered} \tag{4}$$

then for $A > (1 - q)^{-1} B$ the function

$$\Omega(x, y) = V(x) + A|y - y^*(x)| \tag{5}$$

will be a Lyapunov function for the reflection T_n in the neighborhood of the point $(x^*, y^*(x^*))$.

Actually, in a sufficiently small neighborhood of the point $(x^*, y^*(x^*))$

$$\begin{gathered} \Omega(\bar{x}, \bar{y}) - \Omega(x, y) = V[f(x, y^*(x)] - V(x) + V[f(x, y)] - \\ - V[f(x, y^*(x)] + A|g(x, y)| - A|y - y^*(x)| < V[f(x, y^*(x)] - \\ - V(x) - [A(1 - q) - B]|y - y^*(x)| < 0, \end{gathered}$$

which it was also required to prove.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be roots of the characteristic equation of the transformation T, continuously differentiable a sufficient number of times, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be different from $\lambda_{m+1}, \ldots, \lambda_n$. Then after a corresponding linear substitution of variables in the neighborhood of the fixed point $x^* = y^* = 0$ the transformation T may be written in the form:

$$\bar{x} = Lx + \Omega(x, y); \qquad \bar{y} = My + \Delta(x, y). \tag{6}$$

where the matrix L has the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the matrix M the roots $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$; $|\Omega|$ and $|\Delta|$ are not lower

than of the second order of smallness relative to $|x|$ and $|y|$. By virtue of theorem 1, if $|\lambda_{m+1}|, \ldots, |\lambda_n| < 1$ and if the question of the stability of the reflection

$$\bar{x} = Lx + \Omega(x, y^*), \tag{7}$$

where y^* in the neighborhood of a fixed point is unambiguously defined by x in accordance with the equation $y^* = My^* + \Delta(x, y^*)$, can be solved with the aid of a continuously differentiable Lyapunov function, then a fixed point of the reflection T is stable when and only when the fixed reflection point of (7) is stable. It follows from this, in particular, that the consideration of the stability of a fixed point in the transformation of an n-dimensional Euclidean space onto itself in the critical cases where there is a single root, equal to plus or minus one, or two complex conjugate roots $e^{i\varphi}$ and $e^{-i\varphi}$ reduces to the same problem as for the reflection of a straight line onto a straight line or, correspondingly, a plane onto a plane.

2. After squaring the reflection T the case $\lambda_1 = -1$ reduces to the case $\lambda_1 = +1$. Therefore it is sufficient to consider the reflection:

$$\bar{x} = x + a_s x^s + O(x^{s+1}) \qquad (a_s \neq 0). \tag{8}$$

If s is odd, the Lyapunov function sought will be $V = -a_s x^2$; if s is even, then this function will be $V = -a_s x^3$. By virtue of this, when s is even, we shall always get instability and, when s is odd, stability if $a_s < 0$ and instability if $a_s > 0$.

3. The case $\lambda_1 = e^{i\varphi}$, $\lambda_2 = e^{-i\varphi}$, where $0 < \varphi < \pi$, reduces, in accordance with what has been said, to an investigation of the stability of a point reflection of a plane onto a plane of the form:

$$\begin{aligned} \bar{u} &= e^{i\varphi}u + au^2 + buv + cv^2 + du^3 + ku^2v + fuv^2 + gv^3 + \ldots; \\ \bar{v} &= e^{-i\varphi}v + c'u^2 + b'uv + a'v^2 + g'u^3 + f'u^2v + k'uv^2 + d'v^3 + \ldots. \end{aligned} \tag{9}$$

It is known that for any $\varepsilon > 0$, however small, it is possible to select integers n and m such that $|n\varphi - 2\pi m\varphi| < \varepsilon$. Now, iterating the transformation (9) n times, we find that

$$|u_n|^2 = |u|^2 + \varepsilon A|u|^4 + nB|u|^4 + C|u|^6, \qquad (10)$$

where the quantities A and C remain outside the dependence on n and

$$B = 2\mathrm{Re}\left\{ kc^{-is} + ab\,\frac{e^{-i\varphi}(1-e^{-i\varphi})}{1-e^{i\varphi}} + bb'\,\frac{1}{e^{i\varphi}-1} - 2cc'\,\frac{1}{1-e^{3i\varphi}} \right\}. \qquad (11)$$

By virtue of (10) the fixed point u = v = 0 is stable when B < 0 and unstable when B > 0. For the sake of brevity we have omitted to form the Lyapunov function for the reflection (9).

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Submitted: 10 Apr. 1959

DETERMINATION OF THE DISPERSION OF A SEGMENT OF A STATIONARY RANDOM FUNCTION FROM ITS SPECTRUM.

Pages 509-510

by A.G. Kislyakov.

This question is encountered in connection with evaluating the minimum signal detectable by means of a weak signal meter [1] or when it is necessary to find the mean-square frequency deviation of a radio-frequency generator for a time T with respect to a known spectrum of frequency fluctuations [2,3] as well as in other problems.

The spread of any random quantity u(t) about its value $\widetilde{u(t)}$, averaged over time T (we consider $\overline{u(t)} = 0$; the horizontal line indicates the mean with respect to an ensemble), may be characterized by the quantity:

$$D_T = \overline{\widehat{[u(t) - \hat{u}(t)]^2}}, \qquad (1)$$

which we shall call the "dispersion over time T". It is easy to see that $D_T \longrightarrow \overline{u^2(t)}$ when $T \longrightarrow \infty$. Separating the expression in brackets and changing, where necessary the order of averaging with respect to the ensemble and to time, we have:

$$D_T = \overline{u^2(t)} - \overline{\widehat{[u(t)]^2}} =$$

$$= \int_0^\infty w(f)\,df - \int_0^\infty (\pi fT)^{-2} \sin^2(\pi fT)\, w(f)\,df,$$

where w(f) is the mean square spectral density of u(t). Introducing a certain effective spectrum $w_T(f)$ (such that $D_T = \int_0^\infty w_T(f)df$), we get

$$w_T(f) = w(f)\,[1 - (\pi fT)^{-2} \sin^2(\pi fT)] + w_0(f). \qquad (2)$$

where $w_0(f)$ is a certain function, for which $\int_0^\infty w_0(f)df = 0$. A more rigorous calculation, not given here, shows that for stationary random processes $w_0(f) \equiv 0$.

Thus, the first two terms on the right-hand side of expression (2) represent an expression for the effective spectrum $w_T(f)$, defining the dispersion of the process u(t) over the time T. Oscillations with frequencies $f < T^{-1}$ enter into $w_T(f)$ with less weight than into the true spectrum w(f) because of the finiteness of the segment of the function u(t). The part of a weighting function is played in (2) by the factor $g(T,f) = 1 - (\pi fT)^{-2} \sin^2(\pi fT)$. In the literature [2,3] a simpler approximation formula is often used for g(T,f).

$$g(T,f) = \begin{cases} 1 \text{ при } f > T^{-1}, \\ 0 \text{ при } f < T^{-1}. \end{cases} \qquad (3)$$

We then discard from the spectrum w(f) all frequencies, less than T^{-1}, which give slow (so-called "one-sided") deviations. However, in certain cases "one-sided" deviations can make a considerable contribution to the dispersion in time T . A criterion for the applicability of approximation (3) is given by the following condition:

$$|k(T)-1| \ll 1, \tag{4}$$

where $k(T) = D_T/D_T^{\infty}$ and $D_T^{\infty} = \int_{T^{-1}}^{\infty} w(f)df$. Whether condition (4) is fulfilled depends both on the value of T and on the form of the spectrum w(f). For a spectrum

$$w_1(f) = [(2\pi f\tau)^2 + 1]^{-1} \tag{5}$$

the function $k_1(T)$ has the form represented by the graph in Fig. 1a. The greatest error connected with the use of the approximate formula (3) occurs in the region $T \ll \tau$ (τ is the correlation time of the process u(t)).

It is of interest to consider one more example illustrating the role of the form of the spectrum w(f).

Let the spectrum of the process u(t) be described by the formula

$$w_2(f) = \begin{cases} Bf_m^{-\alpha} & \text{при } 0 \leqslant f \leqslant f_m, \\ Bf^{-\alpha} & \text{при } f \geqslant f_m, \end{cases} \tag{6}$$

where $1 < \alpha < 3$. Then*

$$k_2(T) \simeq \pi^{\alpha-1}(\alpha-1)\left\{p(\alpha) + q(\alpha)\varepsilon^{3-\alpha}\right\},$$

where

$$p(\alpha) = \frac{\pi 2^{\alpha-1}}{\alpha(\alpha^2-1)\Gamma(\alpha-1)} \operatorname{cosec}\frac{(\alpha-1)\pi}{2},$$

and $q(\alpha)$ is a certain rational sub-function of α. In the approximation in question the function $k_2(T)$ depends only very slightly on T and the error will be small if we assume that

$$k_2(T) \simeq (\alpha-1)\pi^{\alpha-1}p(\alpha) = k_2(\alpha). \tag{7}$$

The result obtained is accurate for $\varepsilon = 0$. The function $k_2(\alpha)$ is represented in Fig. 1b. As follows from the graph, using the approximate formula (3), which is ordinarily advantageous in simplifying the calculations, does not give a significant error** only when $\alpha \sim 1$.

---

* It is assumed that $\varepsilon = \pi f_m T \ll 1$ and all the terms starting from $\varepsilon^{3-\alpha}$ are neglected.

** It is possible to show that the error connected with using (3) is small for $\alpha < 1$ also, this, moreover, being physically evident.

Physically this means that for a sufficiently slow increase in w(f) with decrease in frequency the "one-sided" deviations make a sufficiently small contribution to the dispersion in time T .

The author wishes to thank V.S. Troitskiy and A.N. Malakhov for useful advice and criticism.

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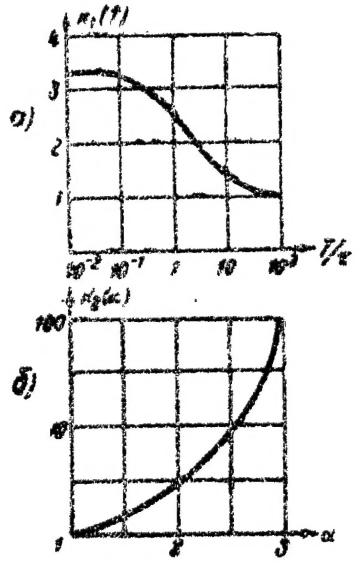

Fig. 1

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ON POSSIBLE WAVES IN A SYSTEM OF PARALLEL ANISOTROPICALLY CONDUCTING PLANES

Pages 510-512

by V.N. Ivanov.

In connection with recent attempts to increase the size of decelerating systems there has been published a number of articles [1,2] devoted to multi-row interdigital decelerating systems. Such systems are designed by employing the theory of multiconductor lines, which considers only the TEM wave and takes inhomogeneities into account by means of equivalent reactances. In order to be able to determine the limits of applicability of such calculations it is necessary to know the rate of attenuation of the higher modes of oscillations in a multiconductor line due to the presence of inhomogeneities.

A rigorous analysis of a multi-row multiconductor line is difficult even in the case of a TEM wave; therefore, for the sake of simplicity, each row of conductors in the multiconductor line will be approximated by an anisotropically conducting plane, with ideal conduction in the direction of the conductors and absence of conduction in the direction at right-angles to the latter. Introducing such an idealization reduces the problem of higher modes of oscillation in a multi-row multiconductor line to the problem of the possible waves in a system of parallel anisotropically conducting planes.

We shall consider a system, consisting of an infinite number of parallel planes a distant d apart, ideally conducting in the y-direction and non-conducting in the z-direction. We shall assume all the quantities to vary in accordance with the law $\exp(i(\omega t - \beta z))$ and number the planes in the direction of increase in x, adopting the numeral 0 for the plane $x = 0$.

The unit component of the vector potential $A_y^{(n)}$, formed by the current in the n-th plate, is equal to

$$A_y^{(n)} = \frac{\mu}{4\pi} \int_{-\infty}^{+\infty} f_n(\eta)\, d\eta \int_{-\infty}^{+\infty} \frac{\exp[-i\beta\zeta] \exp\left[-ik\sqrt{(x-nd)^2+(y-\eta)^2+(z-\zeta)^2}\right]}{\sqrt{(x-nd)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\zeta, \quad (1)$$

where $k = \omega\sqrt{\varepsilon\mu}$, ε and μ are the dielectric constant and the permeability of the medium respectively, $f_n(y) \exp(-i\beta z)$ is the surface current density in the n-th plane, the factor $\exp(i\omega t)$ is omitted, as it will be henceforth. We shall assume that on being displaced by a period d in the x-direction the field acquires a factor

$e^{-i\beta z}$. Then, evaluating the integral with respect to ζ [3] and summing (1) with respect to n from $-\infty$ to $+\infty$, we get the component of the vector potential A_y formed by all the currents:

$$A_y = \frac{\mu e^{-i\beta z}}{2\pi} \int_{-\infty}^{+\infty} f_0(\eta) \left\{ \sum_{n=-\infty}^{+\infty} e^{-in\theta} K_0 \left[\gamma \sqrt{(x-nd)^2 + (y-\eta)^2} \right] \right\} d\eta,$$

where K_0 is a second-order modified Bessel function and $\gamma^2 = \beta^2 - k^2$.

By computing the component of the electric field E_y and requiring it to vanish at all the anisotropically conducting planes, we get the following integral-differential equation for the current density:

$$\left(\frac{\partial^2}{\partial y^2} + k^2 \right) \int_{-\infty}^{+\infty} f_0(\eta) \left\{ \sum_{n=-\infty}^{+\infty} e^{-in\theta} K_0 \left[\gamma \sqrt{n^2 d^2 + (y-\eta)^2} \right] \right\} d\eta = 0. \qquad (2)$$

The theory of homogeneous integral equations, presented in [4], can easily be modified to apply to the case of our integral-differential equation:

$$\left(\frac{\partial^2}{\partial y^2} + k^2 \right) \int_{-\infty}^{+\infty} f(\eta) \, l(y-\eta) = 0. \qquad (3)$$

The possible solutions of (3) have the form:

$$f(y) = C_s e^{i w_s y},$$

where C_s are certain constants determined from the conditions of excitation and w_s zeros of the function

$$L(w) = (k^2 - w^2) \int_{-\infty}^{+\infty} e^{iwy} l(|y|) \, dy.$$

In the case of equation (2), evaluating the integral with respect to y [3], we get:

$$L(w) = \pi \frac{k^2 - w^2}{\sqrt{\gamma^2 + w^2}} \cdot \frac{1 - e^{-2d\sqrt{\gamma^2+w^2}}}{\left[1 - e^{-\left(d\sqrt{\gamma^2+w^2} + i\theta\right)}\right] \left[1 - e^{-\left(d\sqrt{\gamma^2+w^2} - i\theta\right)}\right]}$$

If a wave is propagated in the z-direction with a velocity less than that of light ($\beta^2 > k^2$), then the function L(w) has two real zeros, $w = \pm k$, corresponding to undamped TEM waves and an infinite number of imaginary zeros $w = \pm i\sqrt{\gamma^2 + (\pi l/d)^2}$ ($l = 1, 2, \ldots$), corresponding to TM waves damped in the y-direction.

In addition to the modes discussed TE waves, not connected with the presence of a current, may be observed in the system of anisotrop-

ically conducting planes. When $\beta^2 > k^2$ they will be damped in the y-direction as $e^{-\varkappa_p y}$, where $\varkappa_p^2 = \gamma^2 + (\vartheta + 2\pi p)^2 d^{-2}$ ($p = 0, \pm 1, \pm 2, \ldots$) by virtue of the surface character of slow waves.

On the basis of the results obtained it is possible to conclude that in a system of parallel anisotropically conducting planes in the case of slow waves the only wave propagated in the y-direction is the TEM and that higher modes are damped in the y-direction not more slowly than exp ($-\gamma y$).

In conclusion it should be noted that these results may also be applied to an anisotropically conducting plane, included between two symmetrically disposed screens.

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LETTER TO THE EDITOR

In No. 1 of the journal "Radiofizika" for 1958 there is an article by A.A. Andronov and G.S. Gorelik, "Radiophysics and the General Dynamics of Machines". I was responsible for preparing the manuscript for the printer. On page 11 of the article there is an authors' footnote, in which there is included a reference to the known "Vyshnegradskiy error". Because of an oversight on my part this footnote passed without comment.

The article by Andronov and Gorelik was written in 1944 but later, in 1949, Andronov himself, together with I.N. Voznesenskiy, established that in fact there is no "Vyshnegradskiy error". This has been discussed in detail by Andronov and Voznesenskiy in commentaries included in the monograph "Theory of Automatic Control", D.K. Maxwell, I.A. Vyshnegradskiy, A. Stodola (publ. AS USSR, 1949).

I would like to thank M.I. Freydlin for drawing my attention to this oversight.

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